EXERCISE 9.1 PAGE: 382

Determine order and degree (if defined) of differential equations given in Exercises 1 to 10

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$

Solution:

The given differential equation is,

$$\frac{d^4y}{dx^4} + \sin(y''') = 0$$

⇒ y'''' + sin (y''') = 0

The highest order derivative present in the differential equation is y'''', so its order is three. Hence, the given differential equation is not a polynomial equation in its derivatives and so, its degree is not defined.

2. y' + 5y = 0

Solution:

The given differential equation is, y' + 5y = 0

The highest order derivative present in the differential equation is y', so its order is one. Therefore, the given differential equation is a polynomial equation in its derivatives. So, its degree is one.

3. $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

Solution:-

The given differential equation is,

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$

The highest order derivative present in the differential equation is $\frac{d^2s}{dt^2}$.

The order is two. Therefore, the given differential equation is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$.

So, its degree is one.

4. $\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$

Solution:-

The given differential equation is,

$$\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

The order is two. Therefore, the given differential equation is not a polynomial.
So, its degree is not defined.

5. $\frac{d^2y}{dx^2}=\cos 3x+\sin 3x$

Solution:-

The given differential equation is,

$$\frac{d^2y}{dx^2}=\cos 3x+\sin 3x$$

$$\Rightarrow \frac{d^2y}{dx^2}-\cos 3x-\sin 3x=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

The order is two. Therefore, the given differential equation is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power is 1.

Therefore, its degree is one.

6. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

Solution:

The given differential equation is, $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is y'''.

The order is three. Therefore, the given differential equation is a polynomial equation in y''', y'' and y'.

Then the power raised to y''' is 2.

Therefore, its degree is two.

7. $y''' + 2y'' + y' = 0$

Solution:

The given differential equation is, $y''' + 2y'' + y' = 0$

The highest order derivative present in the differential equation is y'''.

The order is three. Therefore, the given differential equation is a polynomial equation in y''', y'' and y'.

Then the power raised to y''' is 1.

Therefore, its degree is one.

8. $y' + y = e^x$

Solution:

The given differential equation is, $y' + y = e^x$

$= y' + y - e^x = 0$

The highest order derivative present in the differential equation is y'.

The order is one. Therefore, the given differential equation is a polynomial equation in y'.

Then the power raised to y' is 1.

Therefore, its degree is one.

9. $y''' + (y')^2 + 2y = 0$

Solution:

The given differential equation is, $y''' + (y')^2 + 2y = 0$

The highest order derivative present in the differential equation is y''.

The order is two. Therefore, the given differential equation is a polynomial equation in y'' and y'.

Then the power raised to y'' is 1.

Therefore, its degree is one.

10. $y''' + 2y' + \sin y = 0$

Solution:-

The given differential equation is, y''' + 2y' + sin y = 0

The highest order derivative present in the differential equation is y''.

The order is two. Therefore, the given differential equation is a polynomial equation in y'' and y'.

Then the power raised to y'' is 1.

Therefore, its degree is one.

11. The degree of the differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0 \text{ is}$$

(A) 3 **(B) 2** **(C) 1** **(D) not defined**

Solution:-

(D) not defined

The given differential equation is,

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

The order is three. Therefore, the given differential equation is not a polynomial.

Therefore, its degree is not defined.

12. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0 \text{ is}$$

(A) 2 **(B) 1** **(C) 0** **(D) not defined**

Solution:-

(A) 2

The given differential equation is,

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

Therefore, its order is two.

EXERCISE 9.2 PAGE: 385

In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

1. y = ex + 1 : y″ – y′ = 0

Solution:-

From the question it is given that $y = e^x + 1$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) \quad \text{... [Equation (i)]}$$

Now, differentiating equation (i) both sides with respect to x, we have,

$$\frac{d}{dx}(y') = \frac{d}{dx}(e^x)$$

$\Rightarrow y'' = e^x$

Then,

Substituting the values of y′ and y″ in the given differential equations, we get,

$y'' - y' = e^x - e^x$ = RHS.

Therefore, the given function is a solution of the given differential equation.

2. $y = x^2 + 2x + C$: y′ – 2x – 2 = 0

Solution:-

From the question it is given that $y = x^2 + 2x + C$

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}(x^2 + 2x + C)$$

$y' = 2x + 2$

Then,

Substituting the values of y′ in the given differential equations, we get,

$= y' - 2x - 2$

$= 2x + 2 - 2x - 2$

$= 0$

= RHS

Therefore, the given function is a solution of the given differential equation.

3. y = cos x + C : y′ + sin x = 0

Solution:-

From the question it is given that y = cos x + C

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}(\cos x + C)$$

y’ = -sinx

Then,

Substituting the values of y’ in the given differential equations, we get,

= y’ + sinx

= - sinx + sinx

= 0

= RHS

Therefore, the given function is a solution of the given differential equation.

4. y = √(1 + x^2): y’ = ((xy)/(1 + x^2))

Solution:-

From the question it is given that y = $\sqrt{1+x^2}$

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$\Rightarrow y' = \frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)$$

By differentiating (1 + x^2) we get,

$$\Rightarrow y' = \frac{2x}{2\sqrt{1+x^2}}$$

On simplifying we get,

$$\Rightarrow y' = \frac{x}{\sqrt{1+x^2}}$$

By multiplying and dividing √(1 + x^2)

$$\Rightarrow y' = \frac{x}{1+x^2} \times \sqrt{1+x^2}$$

Substituting the value of √(1 + x^2)

Substituting the value of √(1 + x^2)

$$\Rightarrow y' = \frac{x}{1+x^2}.y$$

$$\Rightarrow y' = \frac{xy}{1+x^2}$$

Therefore, LHS = RHS

Therefore, the given function is a solution of the given differential equation.

5. y = Ax : xy' = y (x ≠ 0)
Solution:-
From the question it is given that y = Ax
Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}(Ax)$$

y' = A
Then,
Substituting the values of y' in the given differential equations, we get,
= xy'
= x × A
= Ax
= Y ... [from the question]
= RHS
Therefore, the given function is a solution of the given differential equation

6. y = x sinx : xy' = y + x (√(x^2 – y^2)) (x ≠ 0 and x>y or x< - y)
Solution:-
From the question it is given that y = xsinx

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}(x\sin x)$$

$$\Rightarrow y' = \sin x\frac{d}{dx}(x) + x.\frac{d}{dx}(\sin x)$$

⇒ y' = sinx + xcosx

Then,

Substituting the values of y′ in the given differential equations, we get,

LHS = xy′ = x(sinx + xcosx)

= xsinx + x^2cosx

From the question substitute y instead of xsinx, we get,

$$= y + x^2.\sqrt{1-\sin^2 x}$$

$$= y + x^2\sqrt{1-\left(\frac{y}{x}\right)^2}$$

$$= y + x\sqrt{(y)^2-(x)^2}$$

= RHS

Therefore, the given function is a solution of the given differential equation

7. xy = logy + C : $y' = \dfrac{y^2}{1-xy}\quad (xy \neq 1)$

Solution:-

From the question it is given that xy = logy + C

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$$\Rightarrow y.\frac{d}{dx}(x) + x.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

On simplifying, we get.

$$\Rightarrow y + xy' = \frac{1}{y}\frac{dy}{dx}$$

By cross multiplication,

$$\Rightarrow y^2 + xyy' = y'$$

$$\Rightarrow (xy-1)y' = -y^2$$

$\Rightarrow y' = \frac{y^2}{1-xy}$

By comparing LHS and RHS

LHS = RHS

Therefore, the given function is the solution of the corresponding differential equation.

8. y – cos y = x : (y sin y + cos y + x) y′ = y

Solution:-

From the question it is given that y – cosy = x

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} - \frac{d}{dx}\cos y = \frac{d}{dx}(x)$$

$$\Rightarrow y' + \sin y . y' = 1$$

$$\Rightarrow y'(1 + \sin y) = 1$$

$$\Rightarrow y' = \frac{1}{1+\sin y}$$

Then,

Substituting the values of y′ in the given differential equations, we get,

Consider LHS = (ysiny + cosy + x)y′

$$= (y\sin y + \cos y + y - \cos y) \times \frac{1}{1+\sin y}$$

$$= y(1 + \sin y) \times \frac{1}{1+\sin y}$$

On simplifying we get,

= y

= RHS

Therefore, the given function is the solution of the corresponding differential equation.

9. $x + y = \tan^{-1}y$: $y^2 y' + y^2 + 1 = 0$

Solution:-

From the question it is given that $x + y = \tan^{-1}y$

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(\tan^{-1}y)$$

$$\Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right]y'$$

By transposing y' to RHS and it becomes – y' and take out y' as common for both, we get,

$$\Rightarrow y'\left[\frac{1}{1+y^2} - 1\right] = 1$$

On simplifying,

$$\Rightarrow y'\left[\frac{1-(1+y^2)}{1+y^2}\right] = 1$$

$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right] = 1$$

$$\Rightarrow y' = \frac{-(1+y^2)}{y^2}$$

Then,

Substituting the values of y' in the given differential equations, we get,

Consider, LHS = $y^2y' + y^2 + 1$

$$= y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1$$

$= -1 - y^2 + y^2 + 1$

$= 0$

= RHS

Therefore, the given function is the solution of the corresponding differential equation.

10. $y = \sqrt{a^2 - x^2}\ x \in (-a, a) : \quad x + y\dfrac{dy}{dx} = 0\ (y \neq 0)$

Solution:-

From the question it is given that $y = \sqrt{a^2 - x^2}$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}} \cdot \frac{d}{dx}(a^2 - x^2)$$

$$= \frac{1}{2\sqrt{a^2 - x^2}}(-2x)$$

$$= \frac{-x}{2\sqrt{a^2 - x^2}}$$

Then,
Substituting the values of y' in the given differential equations, we get,

Consider LHS = $x + y\frac{dy}{dx}$

$$= x + \sqrt{a^2 - x^2} \times \frac{-x}{2\sqrt{a^2 - x^2}}$$

On simplifying, we get,

= x −x

= 0

By comparing LHS and RHS

LHS = RHS.

Therefore, the given function is the solution of the corresponding differential equation.

11. The number of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 **(B) 2** **(C) 3** **(D) 4**

Solution:-

(D) 4

The solution which contains arbitrary constants is called the general solution (primitive) of the differential equation.

12. The number of arbitrary constants in the particular solution of a differential equation of third order are:

(A) 3 (B) 2 (C) 1 (D) 0

Solution:-

(D) 0

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution of the differential equation.

EXERCISE 9.3 PAGE: 391

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a}+\frac{y}{b}=1$

Solution:-

From the question it is given that $\frac{x}{a}+\frac{y}{b}=1$

Differentiating both sides with respect to x, we get,

$$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0 \quad \text{... [Equation (i)]}$$

Now, differentiating equation (i) both sides with respect to x, we get,

$$0+\frac{1}{b}y''=0$$

$$\Rightarrow \frac{1}{b}y''=0$$

By cross multiplication, we get,

$\Rightarrow y'' = 0$

∴the required differential equation is $y'' = 0$.

2. $y^2 = a(b^2 - x^2)$

Solution:-

From the question it is given that $y^2 = a(b^2 - x^2)$

Differentiating both sides with respect to x, we get,

$$2y\frac{dy}{dx}=a(-2x)$$

$\Rightarrow 2yy' = -2ax$

$\Rightarrow yy' = (-2/2)ax$

Now, differentiating equation (i) both sides, we get,

$\Rightarrow y' \times y' + yy'' = -a$

$\Rightarrow (y')^2 + yy'' = -a$ ...[we call it as equation (ii)]

Then,

Dividing equation (ii) by (i), we get,

$$\frac{(y')^2 + yy''}{yy'} = \frac{-a}{-ax}$$

$\Rightarrow x(y')^2 + xyy'' = yy'$

Transposing yy' to LHS it becomes – yy'

$\Rightarrow xyy'' + x(y')^2 - yy' = 0$

∴ the required differential equation is $xyy'' + x(y')^2 - yy' = 0$.

3. $y = ae^{3x} + be^{-2x}$

Solution:-

From the question it is given that $y = ae^{3x} + be^{-2x}$ ... [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$y' = 3ae^{3x} - 2be^{-2x}$ ... [equation (ii)]

Now, differentiating equation (ii) both sides, we get,

$y'' = 9ae^{3x} + 4be^{-2x}$ ... [equation (iii)]

Then, multiply equation (i) by 2 and afterwards add it to equation (ii),

We have,

$\Rightarrow (2ae^{3x} + 2be^{-2x}) + (3ae^{3x} - 2be^{-2x}) = 2y + y'$

$\Rightarrow 5ae^{3x} = 2y + y'$

$$\Rightarrow ae^{3x} = \frac{2y + y'}{5}$$

So now, let us multiply equation (ii) by 3 and subtracting equation (ii), We have

$\Rightarrow (3ae^{3x} + 3be^{-2x}) - (3ae^{3x} - 2be^{-2x}) = 3y - y'$

$\Rightarrow 5be^{-2x} = 3y - y'$

$$\Rightarrow be^{-2x} = \frac{3y - y'}{5}$$

Substitute the value of ae^{3x} and be^{-2x} in y'',

$y'' = 9 \times \frac{2y+y'}{5} + 4 \times \frac{2y+y'}{5}$

$$\Rightarrow y'' = \frac{18y + 9y'}{5} + \frac{12y - 4y'}{5}$$

On simplifying we get,

$$\Rightarrow y'' = \frac{30y + 5y'}{5}$$

$\Rightarrow y'' = 6y + y'$

$\Rightarrow y'' - y' - 6y = 0$

$\therefore$the required differential equation is $y'' - y' - 6y = 0$.

4. $y = e^{2x}(a + bx)$

Solution:-

From the question it is given that $y = e^{2x}(a + b\,x)$ … [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$y' = 2e^{2x}(a + b\,x) + e^{2x} \times b$ … [equation (ii)]

Then, multiply equation (i) by 2 and afterwards subtract it to equation (ii),

We have,

$y' - 2y = e^{2x}(2a + 2bx + b) - e^{2x}(2a + 2bx)$

$y' - 2y = 2ae^{2x} + 2e^{2x}bx + e^{2x}b - 2ae^{2x} - 2bxe^{2x}$

$y' - 2y = be^{2x}$ … [equation (iii)]

Now, differentiating equation (iii) both sides,

We have,

$\Rightarrow y'' - 2y = 2be^{2x}$ … [equation (iv)]

Then,

Dividing equation (iv) by (iii), we get,

$$\frac{y'' - 2y'}{y' - 2y} = 2$$

By cross multiplication,

$$\Rightarrow y'' - 2y' = 2y' - 4y$$

Transposing 2y' and -4y to LHS it becomes – 2y' and 4y

$$\Rightarrow y'' - 4y' - 4y = 0$$

∴ the required differential equation is $y'' - 4y' - 4y = 0$.

5. $y = e^x$ (a cos x + b sin x)

Solution:

From the question it is given that $y = e^x(a \cos x + b \sin x)$

… [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$\Rightarrow y' = e^x(a \cos x + b \sin x) + e^x(-a \sin x + b \cos x)$

$\Rightarrow y' = e^x[(a + b)\cos x - (a - b) \sin x)]$ … [equation (ii)]

Now, differentiating equation (ii) both sides,

We have,

$y'' = e^x[(a + b) \cos x - (a - b)\sin x)] + e^x[-(a + b)\sin x - (a - b) \cos x)]$

On simplifying, we get,

$\Rightarrow y'' = e^x[2b\cos x - 2a\sin x]$

$\Rightarrow y'' = 2e^x(b \cos x - a \sin x)$ … [equation (iii)]

Now, adding equation (i) and (iii), we get,

$$y + \frac{y''}{2} = e^x[(a + b)\cos x - (a - b)\sin x]$$

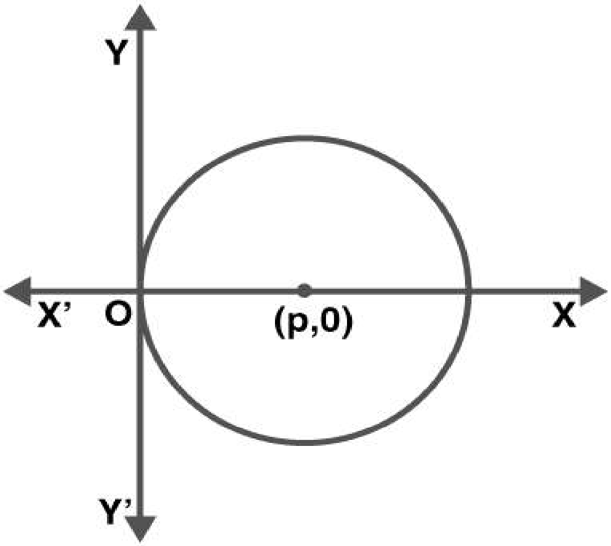
$$y + \frac{y''}{2} = y'$$

$$\Rightarrow 2y + y'' = 2y'$$

Therefore, the required differential equation is $2y + y'' = 2y' = 0$.

6. Form the differential equation of the family of circles touching the y-axis at origin.

Solution:



By looking at the figure we can say that the center of the circle touching the y- axis at origin lies on the x – axis.
Let us assume (p, 0) be the centre of the circle.
Hence, it touches the y – axis at origin, its radius is p.
Now, the equation of the circle with centre (p, 0) and radius (p) is

$\Rightarrow (x - p)^2 + y^2 = p^2$

$\Rightarrow x^2 + p^2 - 2xp + y^2 = p^2$

Transposing p^2 and $-2xp$ to RHS then it becomes $-p^2$ and $2xp$

$\Rightarrow x^2 + y^2 = p^2 - p^2 + 2px$

$\Rightarrow x^2 + y^2 = 2px$ ... [equation (i)]

Now, differentiating equation (i) both sides,
We have,

$\Rightarrow 2x + 2yy' = 2p$

$\Rightarrow x + yy' = p$

Now, on substituting the value of 'p' in the equation, we get,

$\Rightarrow x^2 + y^2 = 2(x + yy')x$

$\Rightarrow 2xyy' + x^2 = y^2$

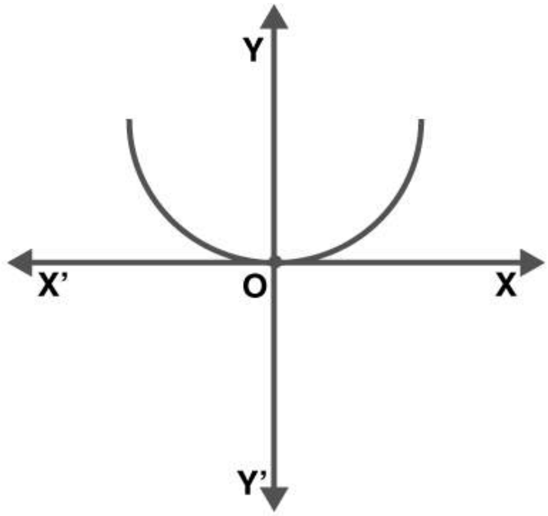
Hence, $2xyy' + x^2 = y^2$ is the required differential equation.

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

Solution:

The parabola having the vertex at origin and the axis along the positive y- axis is

$x^2 = 4ay$ ... [equation (i)



Now, differentiating equation (i) both sides with respect to x,

$2x = 4ay'$ ... [equation (ii)]

Dividing equation (ii) by equation (i), we get,

$$\Rightarrow \frac{2x}{x^2} = \frac{4ay'}{4ay}$$

On simplifying, we get,

$$\Rightarrow \frac{2}{x} = \frac{y'}{y}$$

By cross multiplication,

$\Rightarrow xy' = 2y$

Transposing 2y to LHS it becomes -2y.

$\Rightarrow xy' - 2y = 0$

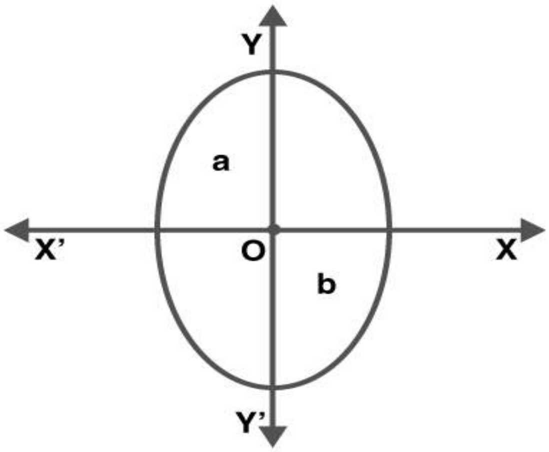
Therefore, the required differential equation is $xy' - 2y = 0$.

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

Solution:

By observing the figure we can say that, the equation of the family of ellipses having foci on y – axis and the centre at origin.

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{... [equation (i)]}$$



Now, differentiating equation (i) both sides with respect to x,

$$\frac{2x}{b^2}+\frac{2yy'}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad \text{... [equation (ii)]}$$

Now, again differentiating equation (ii) both sides with respect to x,

$$\frac{1}{b^2}+\frac{y'y'+yy''}{a^2}=0$$

On simplifying,

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}(y'^2+yy'')=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}(y'^2+yy'')$$

Now substitute the value in equation (ii), we get,

$$x\left[-\frac{1}{a^2}(y'^2+yy'')\right]+\frac{yy'}{a^2}=0$$

On simplifying,

$\Rightarrow -x (y')^2 - xyy'' + yy' = 0$

$\Rightarrow xyy'' + x (y')^2 - yy' = 0$

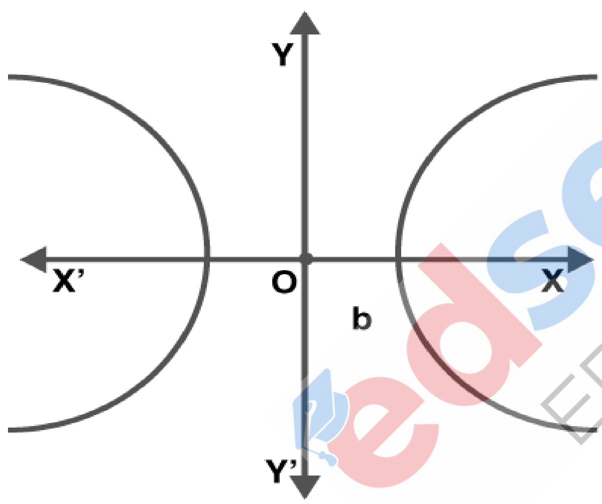
Hence, $xyy'' + x (y')^2 - yy' = 0$ is the required differential equation.

9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

Solution:

By observing the figure we can say that, the equation of the family of hyperbolas having foci on x – axis and the centre at origin is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... [equation (i)]



Now, differentiating equation (i) both sides with respect to x,

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$\Rightarrow \frac{x}{a^2} - \frac{yy'}{b^2} = 0 \qquad \text{... [equation (ii)]}$$

Now, again differentiating equation (ii) both sides with respect to x,

$$\frac{1}{a^2} - \frac{y'y' + yy"}{b^2} = 0$$

On simplifying,

$$\Rightarrow \frac{1}{a^2} - \frac{1}{b^2}(y'^2 + yy") = 0$$

$$\Rightarrow \frac{1}{a^2} = \frac{1}{b^2}(y'^2 + yy")$$

Now substitute the value in equation (ii), we get,

$$\frac{x}{b^2}((y'^2 + yy") - \frac{yy'}{b^2} = 0$$

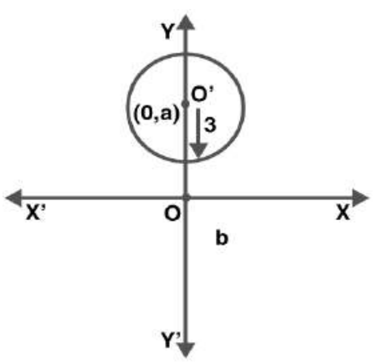
$\Rightarrow x\,(y')^2 + xyy'' - yy' = 0$

$\Rightarrow xyy'' + x(y')^2 - yy' = 0$

Hence, $xyy'' + x\,(y')^2 - yy' = 0$is the required differential equation.

10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

Solution:



Let us assume the centre of the circle on y – axis be (0, a).
We know that the differential equation of the family of circles with centre at (0, a) and radius 3 is: $x^2 + (y- a)^2 = 3^2$

$\Rightarrow x^2 + (y- a)^2 = 9$ … [equation (i)]

Now, differentiating equation (i) both sides with respect to x,

$\Rightarrow 2x + 2(y – a) \times y' = 0$ … [dividing both side by 2]

$\Rightarrow x + (y - a) \times y' = 0$

Transposing x to the RHS it becomes – x.

$\Rightarrow (y – a) \times y' = x$

$\Rightarrow y – a = \frac{-x}{y'}$

Now, substitute the value of (y - a) in equation (i), we get,

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$

Take out the x^2 as common,

$$\Rightarrow x^2\left[1 + \frac{1}{(y')^2}\right] = 9$$

On simplifying,

$\Rightarrow x^2((y')^2 + 1) = 9(y')^2$

$\Rightarrow (x^2 – 9)(y')^2 + x^2 = 0$

Hence, $(x^2 – 9)(y')^2 + x^2 = 0$ is the required differential equation.

11. Which of the following differential equations has $y = c_1 e^x + c^2 e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$ (C) $\frac{d^2y}{dx^2} + 1 = 0$ (D) $\frac{d^2y}{dx^2} - 1 = 0$

Solution:

(B) $\frac{d^2y}{dx^2} - y = 0$

Explanation:
From the question it is given that $y = c_1e^x + c_2e^{-x}$

Now, differentiating given equation both sides with respect to x,

$$\frac{dy}{dx} = c_1e^x - c_2e^{-x} \qquad \text{... [equation (i)]}$$

Now, again differentiating equation (i) both sides with respect to x,

$$\frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

Hence, $\frac{d^2y}{dx^2} - y = 0$ is the required differential equation.

12. Which of the following differential equations has y = x as one of its particular solution?

(A) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$ (B) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

(C) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$ (D) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Solution:

(C) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

Explanation:

From the question it is given that y = x

Now, differentiating given equation both sides with respect to x,

$$\frac{dy}{dx} = 1 \qquad \text{... [equation (i)]}$$

Now, again differentiating equation (i) both sides with respect to x,

$$\frac{d^2y}{dx^2} = 0$$

Then,

Substitute the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given options,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy$$

$= 0 – (x^2 \times 1) + (x \times x)$

$= -x^2 + x^2$

$= 0$

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

We know that 1 – cos x = 2 $\sin^2$ (x/2) and 1 + cos x = 2 $\cos^2$ (x/2)

Using this formula in above function we get

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

We have sin x/cos x = tan x using this we get

$$\Rightarrow \frac{dy}{dx} = \tan^2\frac{x}{2}$$

From the identity $\tan^2 x = \text{sex}^2\ x - 1$, the above equation can be written as

$$\Rightarrow \frac{dy}{dx} = (\sec^2\frac{x}{2} - 1)$$

Now by rearranging and taking integrals on both sides we get

$$\Rightarrow \int dy = \int \sec^2\frac{x}{2}\,dx - \int dx$$

On integrating we get

$$\Rightarrow y = 2\tan^1\frac{x}{2} - x + c$$

2. $\frac{dy}{dx} = \sqrt{4 - y^2}\ (-2 < y < 2)$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sqrt{4 - y^2}$$

On rearranging we get

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}} = dx$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{\sqrt{4-y^2}} = \int dx$$

We know that,

$$\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}$$

Then above equation becomes

$$\Rightarrow \sin^{-1}\frac{y}{2} = x + c$$

3. $\frac{dy}{dx} + y = 1 \ (y \neq 1)$

Solution:

$$\Rightarrow \frac{dy}{dx} + y = 1$$

On rearranging we get

$\Rightarrow$ dy = (1 - y) dx

Separating variables by variable separable method we get

$$\Rightarrow \frac{dy}{1-y} = dx$$

Now by taking integrals on both sides we get

$$\Rightarrow \int \frac{dy}{1-y} = \int dx$$

On integrating

$\Rightarrow$ - log (1 - y) = x + log c

$\Rightarrow$ - log (1 - y) – log c = x

$\Rightarrow$ log (1 - y) c = -x

$$\Rightarrow (1-y)c = e^{-x}$$

Above equation can be written as

$\Rightarrow (1-y) = \frac{1}{c}e^{-x}$

$y = 1 + \frac{1}{c}e^{-x}$

$Y = 1 + A\,e^{-x}$

4. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution:

Given

$\Rightarrow \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$

Dividing both sides by (tan x) (tan y) we get

$\therefore \frac{\sec^2 x \tan y\, dx}{\tan x \tan y} + \frac{\sec^2 y \tan x\, dy}{\tan x \tan y} = 0$

On simplification we get

$\Rightarrow \frac{\sec^2 x\, dx}{\tan x} + \frac{\sec^2 y\, dy}{\tan y} = 0$

Integrating both sides,

$\Rightarrow \int \frac{\sec^2 x\, dx}{\tan x} = \int \frac{\sec^2 y\, dy}{\tan y}$

$\Rightarrow$ let tan x = t &tan y = u

Then

$\sec^2 x\, dx = dt \,\&\, \sec^2 y\, dy = du$

By substituting these in above equation we get

$\therefore \int \frac{dt}{t} = -\int \frac{du}{u}$

On integrating

$\Rightarrow$ log t = -log u + log c

Or,

$\Rightarrow$ log (tan x) = -log (tan y) + log c

$\Rightarrow \log \tan x = \log \frac{c}{\tan y}$

$\Rightarrow$ (tan x) (tan y) = c

5. $(e^x + e^{-x})\, dy - (e^x - e^{-x})\, dx = 0$

Solution:

Given

$\Rightarrow (e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

On rearranging the above equation we get

$$\Rightarrow dy = \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Taking Integrals both sides,

$$\Rightarrow \int dy = \int \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Now let $(e^x + e^{-x}) = t$

Then, $(e^x - e^{-x})dx = dt$

$$\therefore y = \int \frac{dt}{t}$$

On integrating

$$\because \int \frac{dx}{x} = \log x$$

So,

$\Rightarrow y = \log t$

Now by substituting the value of t we get

$\Rightarrow y = \log(e^x + e^{-x}) + C$

6. $\dfrac{dy}{dx} = (1 + x^2)(1 + y^2)$

Solution:

$$\Rightarrow \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{1 + y^2} = dx(1 + x^2)$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{1+y^2} = \int dx + \int x^2 dx$$

On integrating we get

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + c$$

7. $y \log y \, dx - x \, dy = 0$

Solution:

Given

$y \log y \, dx - x \, dy = 0$

On rearranging we get

$\Rightarrow (y \log y) \, dx = x \, dy$

Separating variables by using variable separable method we get

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y \log y}$$

Now integrals on both sides,

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y \log y}$$

$\Rightarrow$ let $\log y = t$

Then

$$\Rightarrow \frac{1}{y} dy = dt$$

$$\Rightarrow \log x = \int \frac{dt}{t}$$

$\Rightarrow \text{Log } x + \log c = \log t$

Now by substituting the value of t

$\Rightarrow \text{Log } x + \log c = \log (\log y)$

Now by using logarithmic formulae we get

$\Rightarrow \text{Log } c x = \log y$

$\Rightarrow \text{Log } y = cx$

$$\Rightarrow y = e^{cx}$$

8. $x^5 \frac{dy}{dx} = -y^5$

Solution:

Given

$$\Rightarrow x^5 \frac{dy}{dx} = -y^5$$

Separating variables by using variable separable method we get

$$\Rightarrow \frac{dy}{y^5} = \frac{-dx}{x^5}$$

On rearranging

$$\Rightarrow \frac{dy}{y^5} + \frac{dx}{x^5} = 0$$

Integrating both sides,

$$\Rightarrow \int \frac{dy}{y^5} + \int \frac{dx}{x^5} = a$$

Let a be a constant,

$$\Rightarrow \int y^{-5} dy + \int x^{-5} dx = a$$

On integrating we get

$$\Rightarrow -4y^{-4} - 4x^{-4} + c = a$$

On simplification we get

$$\Rightarrow -x^{-4} - y^{-4} = c$$

The above equation can be written as

$$\Rightarrow \frac{1}{x^4} + \frac{1}{y^4} = c$$

9. $\frac{dy}{dx} = \sin^{-1} x$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x$$

Separating variables by using variable separable method we get

$\Rightarrow\ dy = \sin^{-1}x\, dx$

Taking integrals on both sides,

$\Rightarrow \int dy = \int \sin^{-1}x\ dx$

Now to integrate $\sin^{-1}x$ we have to multiply it by 1

to use product rule

$\int u.v\, dx = u\int v\ dx - \int\left(\frac{d}{dx}u\right)(\int vdx)dx\}$

Then we get

$\Rightarrow\ y = \int 1.\sin^{-1}x\, dx$

According to product rule and ILATE rule, the above equation can be written as

$\therefore y = \{\sin^{-1}x\int 1.dx - \int(\frac{d}{dx}\sin^{-1}x)(\int 1.dx)dx\}$

On integrating we get

$\Rightarrow\ y = x\sin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx$

Now

$\Rightarrow$ let $1 - x^2 = t$

Then

$\Rightarrow\ -2x\, dx = dt$

$\Rightarrow\ xdx = -\frac{dt}{2}$

Substituting these in above equation we get

$\Rightarrow\ y = x\sin^{-1}x + \int\frac{1}{2\sqrt{t}}dt$

On simplification above equation can be written as

$\Rightarrow\ y = x\sin^{-1}x + \frac{1}{2}\int t^{-\frac{1}{2}}dt$

$\Rightarrow\ y = x\sin^{-1}x + \frac{1}{2}\sqrt{t} + c$

Substituting the value of t, we get

$\Rightarrow\ y = x\sin^{-1}x + \sqrt{1-x^2} + c$

10. $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

Solution:

Given

$\Rightarrow\ e^x \tan y\, dx\ +\ 1(1 - e^x) \sec^2 y\, dy\ =\ 0$

On rearranging above equation can be written as

$\Rightarrow\ (1 - e^x) \sec^2 y\, dy\ =\ -e^x \tan y\, dy\ =\ 0$

Separating the variables by using variable separable method,

$$\Rightarrow\ \frac{\sec^2 y}{\tan\ y} dy\ =\ -\frac{e^x}{1 - e^x} dx$$

Now by taking integrals on both sides, we get

$$\Rightarrow\ \int \frac{\sec^2 y}{\tan y} dy\ =\ \int \frac{e^{-x}}{1 - e^x} dx$$

Let tan y = t and $1 - e^x = u$

Then on differentiating

$(\sec^2 y\, dy\ =\ dt) \& (e^x dx\ =\ du)$

Substituting these in above equation we get

$$\therefore \int \frac{dt}{t} = \int \frac{du}{u}$$

On integrating we get

$\Rightarrow$ Log t = log u + log c

Substituting the values of t and u on above equation.

$\Rightarrow\ \log(\tan y)\ =\ \log(1 - e^x)\ +\ \log c$

$\Rightarrow\ \log \tan y\ =\ \log c(1 - e^x)$

By using logarithmic formula above equation can be written as

$\Rightarrow\ \tan y\ =\ c(1 - e^x)$

For each of the differential equations in Exercises 11 to 14, find a particular solution Satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x;\ y = 1$ when $x = 0$

Solution:

Given

$$\Rightarrow (x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$$

Separating variables by using variable separable method,

$$\Rightarrow dy = \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx$$

Taking integrals on both sides, we get

$$\Rightarrow \int dy = \int \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx \quad \ldots\ldots 1$$

Integrating it partially using partial fraction method,

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{Ax^2 + A(Bx + C)(x + 1)}{(x + 1)(x^2 + 1)}$$

$$\Rightarrow 2x^2 + x = Ax^2 + A + Bx + Cx + C$$

$$\Rightarrow 2x^2 + x = (A + B)x^2 + (B + C)x + A + C$$

Now comparing the coefficients of x^2 and x

$\Rightarrow A + B = 2$

$\Rightarrow B + C = 1$

$\Rightarrow A + C = 0$

Solving them we will get the values of A, B, C

$$A = \frac{1}{2}, B = \frac{3}{2}, C = -\frac{1}{2}$$

Putting the values of A, B, C in 1 we get

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{1}{2}\frac{1}{(x + 1)} + \frac{1}{2}\frac{3x - 1}{x^2 + 1}$$

Now taking integrals on both sides

$$\Rightarrow \int dy = \frac{1}{2}\int \frac{1}{x + 1}dx + \frac{1}{2}\int \frac{3x - 1}{x^2 + 1}dx$$

On integrating

$$\Rightarrow y = \frac{1}{2}\log(x + 1) + \frac{3}{2}\int \frac{x}{x^2 + 1}dx - \frac{1}{2}\int \frac{dx}{x^2 + 1}$$

$$\Rightarrow y = \frac{1}{2}\log(x + 1) + \frac{3}{4}\int \frac{2x}{x^2 + 1}dx - \frac{1}{2}\tan^{-1}x \quad \ldots.2$$

For second term

let $x^2 + 1 = t$

Then, 2x dx = dt

$$\therefore \frac{3}{4}\int \frac{2x}{x^2 + 1}dx = \frac{3}{4}\int \frac{dt}{t}$$

$$\text{so, } I = \frac{3}{4}\log t$$

Substituting the value of t we get

$$I = \frac{3}{4}\log(x^2 + 1)$$

Then 2 becomes

$$\Rightarrow y = \frac{1}{2}\log(x + 1) + \frac{3}{4}\log(x^2 + 1) - \frac{1}{2}\tan^{-1}x + c$$

Taking 4 common

$$\Rightarrow y = \frac{1}{4}[2\log(x + 1) + 3\log(x^2 + 1)] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log(x + 1)^2 + \log(x^2 + 1)^3] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log\{(x + 1)^2 (x^2 + 1)^3\}] - \frac{1}{2}\tan^{-1}x + c \quad \text{....3}$$

Now, we are given that y = 1 when x = 0

$$\therefore 1 = \frac{1}{4}[\log\{(0 + 1)^2 (0^2 + 1)\}] - \frac{1}{2}\tan^{-1}0 + c$$

$$1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + c$$

Therefore,

C = 1

Putting the value of c in 3 we get

$$y = \frac{1}{4}[\log\{(x + 1)^2 (x^2 + 1)^3\}] - \frac{1}{2}\tan^{-1}x + 1$$

12. $x(x^2 - 1)\frac{dy}{dx} = 1$; $y = 0$ when $x = 2$

Solution:

Given

$$x(x^2 + 1)\frac{dy}{dx} = 1$$

Separating variables by variable separable method,

$$\Rightarrow dy = \frac{dx}{x(x^2 + 1)}$$

$X^2 + 1$ can be written as $(x + 1)(x - 1)$ we get

$$\Rightarrow dy = \frac{dx}{x(x + 1)(x - 1)}$$

Taking integrals on both sides,

$$\Rightarrow \int dy = \int \frac{dx}{x(x + 1)(x - 1)} \quad .....1$$

Now by using partial fraction method,

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{c}{x - 1} \quad .....2$$

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A(x - 1)(x + 1) + B(x)(x - 1) + C(x)(x + 1)}{x(x + 1)(x - 1)}$$

Or

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{(A + B + C)x^2 + (B - C)x - A}{x(x + 1)(x - 1)}$$

Now comparing the values of A, B, C

$A + B + C = 0$

$B - C = 0$

$A = -1$

Solving these we will get that $B = ½$ and $C = ½$

Now putting the values of A, B, C in 2

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x + 1}\right) + \frac{1}{2}\left(\frac{1}{x - 1}\right)$$

Now taking integrals we get

$$\Rightarrow \int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \left(\frac{1}{x + 1}\right)dx + \frac{1}{2}\int \left(\frac{1}{x - 1}\right)dx$$

On integrating

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x + 1) + \frac{1}{2}\log(x - 1) + \log c$$

$$\Rightarrow y = \frac{1}{2}\log\left[\frac{c^2(x-1)(x+1)}{x^2}\right\} \quad .....3$$

Now we are given that y = 0 when x = 2

$$0 = \frac{1}{2}\log\left[\frac{c^2(2-1)(2+1)}{4}\right\}$$

$$\Rightarrow \log\frac{3c^2}{4} = 0$$

We know $e^0 = 1$ by substituting we get

$$\Rightarrow \frac{3c^2}{4} = 1$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c^2 = 4/3$$

Now putting the value of c^2 in 3

Then,

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

13. $\cos\left(\frac{dy}{dx}\right) = a \ (a \in \mathbf{R});\ y = 1$ when $x = 0$

Solution:

Given

$$\cos\left(\frac{dy}{dx}\right) = a$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

dy = $\cos^{-1}$ a dx

Integrating both sides, we get

$$\int dy = \cos^{-1} a \int dx$$

$y = x\cos^{-1} a + C$ 1

Now y = 1 when x = 0

Then

$1 = 0\cos^{-1} a + C$

Hence C = 1

Substituting C = 1 in equation (1), we get:

$y = x\cos^{-1} a + 1$

$(y - 1)/x = \cos^{-1} a$

$$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$$

14. $\dfrac{dy}{dx} = y\tan x\,;\ y = 1$ when $x = 0$

Solution:

Given

$$\frac{dy}{dx} = y\tan x$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{y} = \tan x\,dx$$

Taking Integrals both sides, we get

$$\Rightarrow \int \frac{dy}{y} = \int \tan x\,dx$$

On integrating

⇒ Log y = -log (cos x) + log c

Using standard trigonometric identity we get

⇒ Log y = log (sec x) + log c

Using logarithmic formula in above equation we get

⇒ Log y = log c (sec x)

⇒ y = c (sec x) 1

Now we are given that y = 1 when x = 0

⇒ 1 = c (sec 0)

⇒ 1 = c × 1

$\Rightarrow c = 1$
Putting the value of c in 1
$\Rightarrow y = \sec x$

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$

Solution:
To find the equation of a curve that passes through point (0, 0) and has differential equation $y' = e^x \sin x$
So, we need to find the general solution of the given differential equation and the put the given point in to find the value of constant.

$$\text{So,} \Rightarrow \frac{dy}{dx} = e^x \sin x$$

Separating variables by variable separable method, we get

$$\Rightarrow dy = e^x \sin x \ dx$$

Integrating both sides,

$$\Rightarrow \int dy = \int e^x \sin x \, dx \quad \text{....1}$$

Now by using product rule we get

$$\int u.v \, dx = u \int v \, dx - \int \{\frac{d}{dx} u \int v \, dx\} dx$$

Now let

$$I = \int e^x \sin x \, dx$$

$$\Rightarrow I = \sin x \int e^x dx - \int (\frac{d}{dx} \sin x \int e^x dx) dx$$

$$\Rightarrow I = e^x \sin x - \int \cos x \, e^x dx$$

Now by integrating we get

$$\Rightarrow I = e^x \sin x - [\cos x \int e^x dx + \int \sin x \, e^x dx]$$

From 1 we have

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

Now on simplifying

$\Rightarrow\ 2I\ =\ e^x \sin x - e^x \cos x$

$\Rightarrow\ 2I\ =\ e^x(\sin x - \cos x)$

$\Rightarrow\ I\ =\ e^x\dfrac{(\sin x\ - \cos x)}{2}$

Substituting I in 1 we get

$\Rightarrow\ y\ =\ e^x\dfrac{(\sin x\ - \cos x)}{2}\ +\ c$ 2

Now we are given that the curve passes through point (0, 0)

$\therefore 0\ =\ e^0\dfrac{(\sin 0\ - \cos 0)}{2}\ +\ c$

$\Rightarrow\ 0\ =\ \dfrac{1(0-1)}{2}\ +\ c$

$\Rightarrow\ c\ =\ \dfrac{1}{2}$

Substituting the value of C in 2

$\Rightarrow\ y\ =\ e^x\dfrac{(\sin x\ - \cos x)}{2}\ +\ \dfrac{1}{2}$

On rearranging

$\Rightarrow\ 2y\ =\ e^x(\sin x - \cos x)\ +\ 1$

Hence

$\Rightarrow\ 2y - 1\ =\ e^x(\sin x - \cos x)$

16. For the differential equation $xy\dfrac{dy}{dx} = (x+2)\,(y+2)$

Find the solution curve passing through the point (1, –1).

Solution:

For this question, we need to find the particular solution at point (1,-1) for the given differential equation.

Given differential equation is

$\Rightarrow\ xy\dfrac{dy}{dx}\ =\ (x\ +\ 2)(y\ +\ 2)$

Separating variables by variable separable method, we get

$\Rightarrow\ \dfrac{y}{y\ +\ 2}dy\ =\ \dfrac{(x\ +\ 2)dx}{x}$

Taking Integrals both sides, we get

$$\Rightarrow \int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx$$

Splitting the integrals

$$\Rightarrow \int dy - 2\int \frac{1}{y+2} dy = \int dx + 2\int \frac{1}{x} dx$$

$$\Rightarrow y - 2\log(y+2) = x + 2\log x + c \quad ...1$$

Now separating like terms on each side,

$$\Rightarrow y - x - c = 2\log x + 2\log(y+2)$$

$$\Rightarrow y - x - c = \log x^2 + \log(y+2)^2$$

Using logarithmic formula we get

$$\Rightarrow y - x - c = \log\{x^2(y+2)^2\} - i)$$

Now we are given that, the curve passes through (1, -1)

Substituting the values of x and y, to find the value of c

$$\Rightarrow -1 - 1 - c = \log\{1(-1+2)^2\}$$

$\Rightarrow$ -2-c = log (1)

We know that log 1 0

$\Rightarrow$ c = -2 + 0

So c = -2

Substituting the value of c in 1

$$y - x - c = \log\{x^2(y+2)^2\}$$

$$y - x + 2 = \log\{x^2(y+2)^2\}$$

17. Find the equation of a curve passing through the point (0, –2) given that at any point (*x*, *y*) on the curve, the product of the slope of its tangent and *y* coordinate of the point is equal to the *x* coordinate of the point.

Solution:

We know that slope of a tangent is = $\frac{dy}{dx}$.

So we are given that the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

$$y\frac{dy}{dx} = x$$

Now separating variables by variable separable method,

$\Rightarrow$ y dy = x dx

Taking integrals both sides,

$$\Rightarrow \int ydy = \int xdx$$

On integrating we get

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c \quad ...1$$

Now the curve passes through (0, -2).

$\therefore$ 4-0 = 2c

$\Rightarrow$ c = 2

Putting the value of c in 1 we get

$$\Rightarrow y^2 - x^2 = 4$$

18. At any point (*x*, *y*) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Solution:

We know that (x, y) is the point of contact of curve and its tangent.

Slope (m1) for line joining (x, y) and (-4, -3) is $\frac{y+3}{x+4}$......1

Also we know that slope of tangent of a curve is $\frac{dy}{dx}$.

$$\therefore \text{slope (m2) of tangent} = \frac{dy}{dx} \quad .....2$$

Now, according to the question, we can write as

(m2) = 2(m1)

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$

Separating variables by variable separable method, we get

$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{y + 3} = 2\int \frac{dx}{x + 4}$$

On integrating we get

$\Rightarrow \log(y + 3) = 2\log(x + 4) + \log c$

Using logarithmic formula above equation can be written as

$$\Rightarrow \log(y + 3) = \log c(x + 4)^2$$

$$\Rightarrow y + 3 = c(x + 4)^2 \quad ....3$$

Now, this equation passes through the point (-2, 1).

$$\Rightarrow 1 + 3 = c(-2 + 4)^2$$

$\Rightarrow 4 = 4c$

$\Rightarrow c = 1$

Substitute the value of c in 3

$$\Rightarrow y + 3 = (x + 4)^2$$

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after *t* seconds.

Solution:

Let the rate of change of the volume of the balloon be k where k is a constant

$$\therefore \frac{dy}{dt} = k$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \left\{\text{volume of sphere} = \frac{4}{3}\pi r^3\right\}$$

On differentiating with respect to r we get

$$\Rightarrow \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = k$$

On rearranging

$$\Rightarrow 4\pi r^2 dr = kdt$$

Taking integrals on both sides,

$$\Rightarrow 4\pi \int r^2 dr = k\int dt$$

On integrating we get

$$\Rightarrow \frac{4\pi r^3}{3} = kt + c \quad ....1$$

Now, from the question we have

At t = 0, r = 3:

$\Rightarrow 4\pi \times 33 = 3(k \times 0 + c)$

$\Rightarrow 108\,\pi = 3c$

$\Rightarrow c = 36\pi$

At t = 3, r = 6:

$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + c)$

$\Rightarrow k = 84\,\pi$

Substituting the values of k and c in 1

$\Rightarrow 4\pi r^3 = 3(84\pi t + 36\pi)$

$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$

$\Rightarrow r^3 = 63t + 27$

$\Rightarrow r = \sqrt[3]{63t + 27}$

So the radius of balloon after t seconds is $\sqrt[3]{63t + 27}$

20. In a bank, principal increases continuously at the rate of *r*% per year. Find the value of *r* if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Solution:

Let t be time, p be principal and r be rate of interest

According the information principal increases at the rate of r% per year.

$$\therefore \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

Separating variables by variable separable method, we get

$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dp}{p} = \frac{r}{100}\int dt$$

On integrating we get

$$\Rightarrow \log p = \frac{rt}{100} + k$$

$$\Rightarrow p = e^{\frac{rt}{100} + k} \quad ....1$$

Given that t = 0, p = 100.

$\Rightarrow\ 100 = e^{k}$ 2

Now, if t = 10, then p = 2 × 100 = 200

So,

$\Rightarrow\ 200 = e^{\frac{rt}{10}+k}$

$\Rightarrow\ 200 = e^{\frac{rt}{10}}.e^{k}$

From 2

$\Rightarrow\ 200 = e^{\frac{rt}{10}} \times 100$

$\Rightarrow\ e^{\frac{r}{10}} = 2$

$\Rightarrow\ \frac{r}{10} = \log 2$

$\Rightarrow\ r = 6.93$

So r is 6.93%.

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5}$ = 1.648).

Solution:

Let p and t be principal and time respectively.

Given that principal increases continuously at rate of 5% per year.

$\therefore \frac{dp}{dt} = \left(\frac{5}{100}\right)p$

Separating variables by variable separable method,

$\Rightarrow\ \frac{dp}{p} = \frac{p}{25}$

Taking integrals on both sides,

$\Rightarrow\ \int \frac{dp}{p} = \frac{1}{20}\int dt$

$\Rightarrow\ \log p = e^{\frac{t}{20}+c}$ ...1

When t = 0, p = 1000

$\Rightarrow\ 1000 = e^{c}$

At t = 10

$\Rightarrow\ p\ =\ e^{\frac{1}{2}+c}$

The above equation can be written as

$\Rightarrow\ p\ =\ e^{0.5}\times e^{c}$

$\Rightarrow\ p\ =\ 1.648\times 1000\ (e^{0.5}\ =\ 1.648)$

$\Rightarrow\ p\ =\ 1648$

So after 10 years the total amount would be Rs.1648

22. In a culture, the bacteria count is 1, 00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2, 00,000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let y be the number of bacteria at any instant t.

Given that the rate of growth of bacteria is proportional to the number present

$\therefore \frac{dy}{dt} \propto y$

$\Rightarrow\ \frac{dy}{dt}\ =\ ky\ (k \text{ is a constant})$

Separating variables by variable separable method we get,

$\Rightarrow\ \frac{dy}{dt}\ =\ kdt$

Taking integrals on both sides,

$\Rightarrow\ \int \frac{dy}{y}\ =\ k\int dt$

On integrating we get

$\Rightarrow$ Log y = k t + c...1

Let y' be the number of bacteria at t = 0.

$\Rightarrow$ Log y' = c

Substituting the value of c in 1

$\Rightarrow$ log y = k t + log y'

$\Rightarrow$ Log y- log y' = k t

Using logarithmic formula we get

$$\Rightarrow \log\frac{y}{y'} = kt \quad ....2$$

Also, given that number of bacteria increases by 10% in 2 hours.

Therefore,

$$\Rightarrow y = \frac{110}{100}y'$$

$$\Rightarrow \frac{y}{y'} = \frac{11}{10} \quad ....3$$

Substituting this value in 2, we get

$$\Rightarrow k \times 2 = \log\frac{11}{10}$$

$$\Rightarrow k = \frac{1}{2}\log\frac{11}{10}$$

So, 2 becomes

$$\Rightarrow \frac{1}{2}\log\frac{11}{10} \times t = \log\frac{y}{y'}$$

$$\Rightarrow t = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}} \quad ...4$$

Now, let the time when number of bacteria increase from 100000 to 200000 be t'.

$$\Rightarrow y = 2y' \text{ at } t = t'$$

So from 4, we have

$$\Rightarrow t' = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}} = \frac{2\log 2}{\log\frac{11}{10}}$$

So bacteria increases from 100000 to 200000 in $\frac{2\log 2}{\log\frac{11}{10}}$ hours.

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Solution:

(A) $e^x + e^{-y} = C$

Explanation:

We have

$$\Rightarrow \frac{dy}{dx} = e^{x+y}$$

Using laws of exponents we get

$$\Rightarrow \frac{dy}{dx} = e^x \times e^y$$

Separating variables by variable separable method we get

$$\Rightarrow e^{-y}dy = e^x dx$$

Now taking integrals on both sides

$$\Rightarrow \int e^{-y}dy = \int e^x dx$$

On integrating

$$\Rightarrow -e^{-y} = e^x + c$$

$$\Rightarrow e^x + e^{-y} = -c$$

Or,

$$e^x + e^{-y} = c$$

So the correct option is A.

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + x y)\, dy = (x^2 + y^2)\, dx$

Solution:

On rearranging the given equation we get

$$\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$$

$$\text{Let } f(x,y) = \frac{x^2+y^2}{x^2+xy}$$

Here, substituting x = k x and y = k y

$$f(kx, ky) = \frac{(kx)^2 + (ky)^2}{(kx)^2 + kx.ky}$$

Taking k^2 common

$$= \frac{k^2}{k^2} . \frac{x^2+y^2}{x^2+xy}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$(x^2 + x y)\, dy = (x^2 + y^2)\, dx$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$$

To solve it we make the substitution.

$y = v x$

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We have dy/dx, substituting this in above equation

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x.vx}$$

Taking x^2 common

$$v + x\frac{dv}{dx} = \frac{x^2(1 + v^2)}{x^2(1 + v)}$$

On simplification we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

On rearranging the above equation we get

$$x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{1 + v^2 - v - v^2}{1 + v}$$

$$x\frac{dv}{dx} = \frac{1 - v}{1 + v}$$

$$\frac{1 + v}{1 - v}dv = \frac{1}{x}dx$$

Taking integrals on both side,

$$\int \frac{1 + v}{1 - v}dv = \int \frac{1}{x}dx$$

$$\int \left(-1 + \frac{2}{1 - v}\right)dv = \int \frac{1}{x}dx$$

On integrating we get

$- v - 2\log|1 - v| = \log|x| + \log c$

Substituting the value of v we get

$$-\frac{y}{x} - 2\log|1 - \frac{y}{x}| = \log|x| + \log C$$

Using logarithmic formula we get

$$-\frac{y}{x} = \log\frac{(x - y)^2}{x^2} + \log|x| + \log C$$

$$-\frac{y}{x} = \log\frac{(x - y)^2}{x^2}.Cx$$

On rearranging and computing we get

$$-\frac{y}{x} = \log\frac{(x - y)^2}{x}C$$

$$\frac{C(x - y)^2}{x} = e^{-y/x}$$

$$C(x - y)^2 = xe^{-y/x}$$

2. $y' = \frac{x+y}{x}$

Solution:

Given

$y' = \frac{x+y}{x}$

The above equation can be written as

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Let $f(x,y) = \frac{x+y}{x}$

Here, putting x = k x and y = k y

$$f(kx, ky) = \frac{kx + ky}{kx}$$

$$= \frac{k}{k} \cdot \frac{x+y}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$y' = \frac{x+y}{x}$$

Then the above equation can be written as

$$\frac{dy}{dx} = \frac{x+y}{x}$$

To solve it we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of v we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

On simplification we get

$$v + x\frac{dv}{dx} = 1 + v$$

On rearranging we get

$$x\frac{dv}{dx} = 1$$

$$dv = \frac{1}{x}dx$$

Now taking integrals on both side we get

$$\int dv = \int \frac{1}{x}dx$$

On integrating we get

v = log x + C

Now by substituting the value of v

$$\frac{y}{x} = \log x + C$$

y = x log x + C x

3. (x – y) dy – (x + y) dx = 0

Solution:

Given (x - y) dy = (x + y) dx

On rearranging above equation we can write as

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$\text{Let } f(x,y) = \frac{x+y}{x-y}$$

Now by substituting x = k x and y = k y

$$f(kx, ky) = \frac{kx+ky}{kx-ky}$$

On simplification we get

$$f(kx, ky) = \frac{x+y}{x-y}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

(x - y) dy – (x + y) dx = 0

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

For further simplification we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dv/dx we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx}$$

Taking x as common we get

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

On rearranging

$$x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v$$

Now taking LCM and computing we get

$$x\frac{dv}{dx} = \frac{1 + v - v + v^2}{1 - v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{1 - v}$$

$$\frac{1 - v}{1 + v^2}dv = \frac{1}{x}dx$$

Taking integrals on both sides we get,

$$\int \frac{1 - v}{1 + v^2}dv = \int \frac{1}{x}dx$$

Now by splitting the integrals we get

$$\int \frac{1}{1 + v^2}dv - \int \frac{v}{1 + v^2}dv = \int \frac{1}{x}dx \quad .....1$$

$$\text{Let, } I_1 = \int \frac{v}{1 + v^2}dv$$

Put $1 + v^2 = t$

2v dv = dt

$$vdv = \frac{1}{2}dt$$

Now by applying integral we get

$$\frac{1}{2}\int \frac{1}{t}dt$$

$\frac{1}{2}\log t$

Now by substituting the value of t we get

$\frac{1}{2}\log(1 + v^2)$

From equation 1 we have

$\therefore \tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log x + C$

Now by substituting the value of v we get

$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log(1 + \left(\frac{y}{x}\right)^2) = \log x + C$

On rearranging we get

$\tan^{-1}\frac{y}{x} = \log x + \frac{1}{2}\log\left(\frac{x^2 + y^2}{x^2}\right) + C$

$\tan^{-1}\frac{y}{x} = \frac{1}{2}\left(2\log x + \log\left(\frac{x^2 + y^2}{x^2}\right)\right) + C$

Using logarithmic formula we get

$\tan^{-1}\frac{y}{x} = \frac{1}{2}\left(\log\left(\frac{x^2 + y^2}{x^2} \times x^2\right)\right) + C$

$\tan^{-1}\frac{y}{x} = \frac{1}{2}(\log x^2 + y^2) + C$

4. $(x^2 - y^2)dx + 2xy\,dy = 0$

Solution:

The given equation can be written as

$2xy\,dy = -(x^2 - y^2)dx$

On rearranging we get

$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$

Let $f(x,y) = -\frac{x^2 - y^2}{2xy}$

Here, substituting x = k x and y = k y

$$f(kx, ky) = -\frac{k^2x^2 - k^2y^2}{2k^2xy}$$

Now by taking k^2 common

$$f(kx, ky) = -\frac{k^2}{k^2}.\frac{x^2 - y^2}{2xy}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$(x^2 - y^2)dx + 2xy\,dy = 0$$

Again on rearranging

$$2xy\,dy = -(x^2 - y^2)dx$$

The above equation can be written as

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

To solve above equation and for further simplification we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{x^2 - v^2x^2}{2x.\,vx}$$

Now taking x^2 as common

$$v + x\frac{dv}{dx} = -\frac{x^2(1 - v^2)}{2vx^2}$$

On rearranging

$$x\frac{dv}{dx} = -\frac{1 - v^2}{2v} - v$$

Now taking LCM and computing

$$x\frac{dv}{dx} = \frac{-1 + v^2 - 2v^2}{2v}$$

On simplification

$$x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$

Rearranging the above equation we get

$$-\frac{2v}{1 + v^2}dv = \frac{1}{x}dx$$

Now by multiplying the above equation by negative sign we get

$$\frac{2v}{1 + v^2}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{2v}{1 + v^2}dv = -\int \frac{1}{x}dx \quad .....1$$

$$\text{Let, } I_1 = \int \frac{2v}{1 + v^2}dv$$

Put $1 + v^2 = t$

$2v\ dv = dt$

$$vdv = \frac{1}{2}dt$$

Taking integral we get

$$\int \frac{1}{t}dt$$

Log t

From 1 we have

$\therefore \log(1 + v^2) = -\log x + \log C$

Now by substituting the value of v we get

$$\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + \log C$$

By using logarithmic formula we get

$$\log\left(\frac{x^2 + y^2}{x^2}\right) = \log\frac{C}{x}$$

On simplification

$x^2 + y^2 = Cx$

5. $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

Solution:

The given question can be written as

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

Let $f(x,y) = \frac{x^2 - 2y^2 + xy}{x^2}$

Now by substituting x = k x and y = k y

$$f(kx, ky) = \frac{k^2x^2 - 2k^2y^2 + kxky}{k^2x^2}$$

Now by taking k^2 common we get

$$f(kx, ky) = \frac{k^2}{k^2} \cdot \frac{x^2 - 2y^2 + xy}{x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$$

On rearranging we get

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

To solve above equation and to make simplification easier we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{x^2 - 2v^2x^2 + x.vx}{x^2}$$

On rearranging we get

$$v + x\frac{dv}{dx} = \frac{1 - 2v^2 + v}{1}$$

$$v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

On simplification

$$x\frac{dv}{dx} = 1 - 2v^2$$

By separating the variables using variable separable method,

$$\frac{1}{1-2v^2}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{1-2v^2}dv = \int \frac{1}{x}dx$$

The above equation can be written as

$$\int \frac{1}{1-(\sqrt{2}v)^2}dv = \int \frac{1}{x}dx$$

$$\int \frac{1}{1^2-(\sqrt{2}v)^2}dv = \int \frac{1}{x}dx$$

On integrating using standard trigonometric identity we get

$$\frac{1}{\sqrt{2}}.\frac{1}{2.1}.\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right| = \log|x| + C$$

Now by substituting the value of v we get

$$\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\frac{y}{x}}{1-\sqrt{2}\frac{y}{x}}\right| = \log|x| + C$$

On simplification

$$\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + C$$

6. $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

Solution:

The given question can be written as

$$xdy = (\sqrt{x^2+y^2} + y)dx$$

On rearranging the above equation we get

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+y^2} + y)}{x}$$

Let $f(x,y) = \frac{(\sqrt{x^2+y^2} + y)}{x}$

Here, putting x = k x and y = k y

$$f(kx, ky) = \frac{(\sqrt{k^2x^2+k^2y^2} + ky)}{kx}$$

Now taking k as common

$$f(kx, ky) = \frac{k}{k}.\frac{(\sqrt{x^2+y^2} + y)}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$xdy - ydx = \sqrt{x^2+y^2}dx$$

By separating the variables using variable separable method we get

$$xdy = (\sqrt{x^2+y^2} + y)dx$$

On rearranging we get

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+y^2} + y)}{x}$$

To solve above equation we make the substitution.

$y = v\,x$

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+x^2v^2} + vx}{x}$$

Taking x as common and computing we get

$$v + x\frac{dv}{dx} = \frac{x\sqrt{1 + v^2} + vx}{x}$$

On simplification

$$v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

Again separating variables we get

$$\frac{1}{\sqrt{1+v^2}}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx$$

Using $\int \frac{1}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2+a^2}$, the above equation can be written as

$$\log(v + \sqrt{1 + v^2}) = \log x + \log C$$

Now by using logarithmic formula we get

$$\log\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = \log Cx$$

On simplifying we get

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx$$

Taking LCM

$$\frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}} = Cx$$

$$\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = Cx$$

On rearranging

$$y + \sqrt{x^2+y^2} = Cx^2$$

7. $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\, dy$

Solution:

The given question can be written as

$$\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\} x}$$

Let $f(x,y) = \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x}$

Now by substituting x = kx and y = ky

$$f(kx, ky) = \frac{\left\{kx\cos\left(\frac{ky}{kx}\right) + ky\sin\left(\frac{ky}{kx}\right)\right\} ky}{\left\{ky\sin\left(\frac{ky}{kx}\right) - kx\cos\left(\frac{ky}{kx}\right)\right\} kx}$$

Now by taking k^2 as common we get

$$f(kx, ky) = \frac{k^2}{k^2}\cdot\frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x}$$

$= k^0.f(x,y)$

Therefore, the given differential equation is homogeneous.

$$\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x}$$

To solve above equation we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting dy/dx value and on rearranging we get

$$v + x\frac{dv}{dx} = \frac{\{x\cos(v) + vx\sin(v)\}vx}{\{vx\sin(v) - x\cos(v)\}x}$$

Taking x as common and simplifying we get

$$v + x\frac{dv}{dx} = \frac{\{\cos(v) + v\sin(v)\}v}{\{v\sin(v) - \cos(v)\}}$$

On rearranging and computing we get

$$x\frac{dv}{dx} = \frac{\{\cos(v) + v\sin(v)\}v}{\{v\sin(v) - \cos(v)\}} - v$$

Taking LCM and simplifying we get

$$x\frac{dv}{dx} = \frac{v\cos(v) + v^2\sin(v) - v^2\sin(v) + v\cos(v)}{v\sin(v) - \cos(v)}$$

$$x\frac{dv}{dx} = \frac{2v\cos(v)}{v\sin(v) - \cos(v)}$$

Separating the variables by using variable separable method we get

$$\frac{v\sin(v) - \cos v}{2v\cos v}dv = \frac{1}{x}dx$$

Now by splitting the numerator we get

$$\frac{v\sin v}{2v\cos v}dv - \frac{\cos v}{2v\cos v}dv = \frac{1}{x}dx$$

On simplification we get

$$\frac{1}{2}\tan v dv - \frac{1}{2}.\frac{1}{v}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\frac{1}{2}\int \tan v dv - \frac{1}{2}.\int \frac{1}{v}dv = \int \frac{1}{x}dx$$

On integrating we get

$$\frac{1}{2}\log\sec v - \frac{1}{2}\log v = \log x + \log k$$

Using logarithmic formula we get

log secv – logv = 2logkx

Now by substituting the value of v we get

$$\log\sec\left(\frac{y}{x}\right) - \log\left(\frac{y}{x}\right) = 2\log kx$$

Ahain using logarithmic formula we gte

$$\log\left(\frac{x}{y}\sec\left(\frac{y}{x}\right)\right) = \log(kx)^2$$

On simplification

$$\frac{x}{y}\sec\left(\frac{y}{x}\right) = k^2x^2$$

We know that sec x = 1/cos x, by using this in above equation we get

$$\frac{1}{xy\cos\left(\frac{y}{x}\right)} = k^2$$

On rearranging

$$xy\cos\left(\frac{y}{x}\right) = \frac{1}{k^2}$$

Where C is integral constant

$$C = \frac{1}{k^2}$$

$$xy\cos\left(\frac{y}{x}\right) = C$$

8. $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

Solution:

The given question can be written as

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

On rearranging we get

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

Let $f(x,y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$

Now put x= k x and y = k y

$$f(kx, ky) = \frac{ky - kx\sin\left(\frac{ky}{kx}\right)}{kx}$$

By taking k as common we get

$$f(kx, ky) = \frac{k}{k} \cdot \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

On rearranging the above equation

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

To solve above equation we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin\left(\frac{vx}{x}\right)}{x}$$

On simplification we get

$$v + x\frac{dv}{dx} = v - \sin v$$

$$x\frac{dv}{dx} = -\sin v$$

Now separating variables by variable separable method we get

$$\frac{1}{\sin v}dv = -\frac{1}{x}dx$$

We know that 1/sin x = cosec x then above equation becomes

$$\operatorname{cosec} v dv = -\frac{1}{x}dx$$

Taking integration on both side, we get

$$\int \operatorname{cosec} v dv = -\int \frac{1}{x}dx$$

On integrating we get

Log (cosec v – cot v) = -log x + log C

Now by substituting the value of v we get

$$\log(\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}) = \log\frac{C}{x}$$

On simplifying we get

$$\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x} = \frac{C}{x}$$

We know that 1/sin x = cosec x and cot x = cos x/ sin x then above equation becomes

$$\frac{1}{\sin\frac{y}{x}} - \frac{\cos\frac{y}{x}}{\sin\frac{y}{x}} = \frac{C}{x}$$

On rearranging we get

$$1 - \cos\frac{y}{x} = \frac{C}{x}.\sin\frac{y}{x}$$

$$x(1 - \cos\frac{y}{x}) = C\sin\frac{y}{x}$$

9. $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$

Solution:

Given

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

The given equation can be written as

$$x\log\left(\frac{y}{x}\right)dy - 2xdy = -ydx$$

Taking dy common

$$\left(x\log\left(\frac{y}{x}\right)dy - 2x\right)dy = -ydx$$

On rearranging we get

$$\frac{dy}{dx} = \frac{-y}{x\log\left(\frac{y}{x}\right)dy - 2x}$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$$\text{Let } f(x,y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Now put x = k x and y = k y

$$f(kx, ky) = \frac{ky}{2kx - kx\log\left(\frac{ky}{kx}\right)}$$

Taking k as common

$$f(kx, ky) = \frac{k}{k}\cdot\frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

$$x\log\left(\frac{y}{x}\right)dy - 2xdy = -ydx$$

On rearranging

$$\frac{dy}{dx} = \frac{-y}{x\log\left(\frac{y}{x}\right)dy - 2x}$$

Simplifying we get

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

To solve it we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting dy/dx value we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

On simplification

$$v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$

$$x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

Taking LCM and simplifying we get

$$x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$x\frac{dv}{dx} = \frac{-v + v\log v}{2 - \log v}$$

By separating the variables using variable separable method we get

$$\frac{2 - \log v}{-v + v\log v}dv = \frac{1}{x}dx$$

$$\frac{2 - \log v}{v(\log v - 1)}dv = \frac{1}{x}dx$$

On simplifying we get

$$\frac{1 - (\log v - 1)}{v(\log v - 1)}dv = \frac{1}{x}dx$$

$$\frac{1}{v(\log v - 1)}dv - \frac{1}{v}dv = \frac{1}{x}dx$$

Integrating both sides, we get $\int \frac{1}{v(\log v - 1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$ ...1

$$\text{Let, } I_1 = \int \frac{1}{v(\log v - 1)} dv$$

Put, log v – 1 = t

$$\frac{1}{v} dv = dt$$

On integrating

$$\int \frac{1}{t} dt$$

Log t

Substituting the value of t

Log (log v - 1)

From equation 1 we have

∴ Log (log v - 1) – log (v) = log (x) + log (c)

By using logarithmic formula we get

$$\log\left(\frac{\log v - 1}{v}\right) = \log(Cx)$$

$$\frac{\log v - 1}{v} = Cx$$

On simplification we get

$$\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}} = Cx$$

$$\frac{x}{y}\left(\log\left(\frac{y}{x}\right) - 1\right) = Cx$$

$$\log\left(\frac{y}{x}\right) - 1 = Cy$$

10. $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Solution:

Given question can be written as

$$\frac{dy}{dx} = \frac{-e^{x/y}\left(1 - \frac{x}{y}\right)}{(1 + e^{x/y})}$$

Let $f(x,y) = \dfrac{-e^{x/y}\left(1-\dfrac{x}{y}\right)}{(1+e^{x/y})}$

Now put x = k x and y = k y

$$f(kx, ky) = \frac{-e^{kx/ky}\left(1-\dfrac{kx}{ky}\right)}{(1+e^{kx/ky})}$$

$$= \frac{-e^{x/y}\left(1-\dfrac{x}{y}\right)}{(1+e^{x/y})}$$

$= k^0 f(x, y)$

Therefore, the given differential equation is homogeneous.

$$\left(1+e^{x/y}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$$

On rearranging

$$\left(1+e^{x/y}\right)dx = -e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy$$

$$\frac{dx}{dy} = \frac{-e^{x/y}\left(1-\dfrac{x}{y}\right)}{(1+e^{x/y})}$$

To solve above equation we make the substitution.

x = v y

Differentiation above equation with respect to x, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

On rearranging and substituting for dy/dx value we get

$$v + y\frac{dv}{dy} = \frac{-e^{vy/y}\left(1-\dfrac{vy}{y}\right)}{(1+e^{vy/y})}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v}+ve^{v}}{1+e^{v}} - v$$

Now taking LCM and simplifying we get

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$$

The above equation can be written as

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^{v}}{1+e^{v}}\right]$$

$$\Rightarrow \left[\frac{1+e^{v}}{v+e^{v}}\right]dv = -\frac{dy}{y}$$

Integrating both sides we get

$$\Rightarrow \log\left(v+e^{v}\right) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

Using logarithmic formula the above equation can be written as

$$\Rightarrow \left[\frac{x}{y}+e^{\frac{x}{y}}\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^{\frac{x}{y}} = C$$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. (x + y) dy + (x – y) dx = 0; y = 1 when x = 1

Solution:

Given

(x + y) dy + (x - y) dx = 0

The above equation can be written as

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)}$$

$$\text{Let } f(x,y) = -\frac{(x-y)}{(x+y)}$$

Now put x= k x and y = k y

$$f(kx, ky) = -\frac{(kx-ky)}{(kx+ky)}$$

By taking k common from both numerator and denominator we get

$$= \frac{k}{k}.-\frac{(x-y)}{(x+y)}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

(x + y) dy + (x - y) dx = 0

Again above equation can be written as

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{(x-vx)}{(x+vx)}$$

Taking x common and simplifying we get

$$v + x\frac{dv}{dx} = -\frac{(1-v)}{(1+v)}$$

On rearranging

$$x\frac{dv}{dx} = -\frac{(1-v)}{(1+v)} - v$$

Taking LCM and simplifying

$$x\frac{dv}{dx} = \frac{-1+v-v-v^2}{(1+v)}$$

$$x\frac{dv}{dx} = \frac{-1-v^2}{(1+v)}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{(1+v)}$$

Then above equation can be written as

$$\frac{1+v}{1+v^2}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1+v}{1+v^2}dv = -\int \frac{1}{x}dx$$

Splitting the denominator,

$$\int \frac{1}{1+v^2}dv + \int \frac{v}{1+v^2}dv = -\int \frac{1}{x}dx$$

On integrating we get

$$\tan^{-1}v + \frac{1}{2}\log(1 + v^2) = -\log x + C$$

Now by substituting the value of v we get

$$\tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + C$$

y = 1 when x = 1

$$\tan^{-1}\frac{1}{1} + \frac{1}{2}\log\left(1 + \left(\frac{1}{1}\right)^2\right) = -\log 1 + C$$

The above equation becomes,

$$\frac{\pi}{4} + \frac{1}{2}\log 2 = 0 + C$$

$$C = \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$\therefore \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + C$$

where, $C = \frac{\pi}{4} + \frac{1}{2}\log 2$

$$\therefore \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right)$$

$$= -\log x + \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$2\tan^{-1}\frac{y}{x} + \log\left(\frac{x^2+y^2}{x^2}\right)$$

$$= -2\log x + \frac{\pi}{2} + \log 2$$

On simplifying we get

$$2\tan^{-1}\frac{y}{x} + \log\left(\frac{x^2+y^2}{x^2}\right) + \log x^2 = \frac{\pi}{2} + \log 2$$

$$2\tan^{-1}\frac{y}{x} + \log(x^2+y^2) = \frac{\pi}{2} + \log 2$$

The required solution of the differential equation.

12. $x^2dy + (x\,y + y^2)dx = 0$; y = 1 when x = 1

Solution:

Given

$x^2dy + (x y + y^2) dx = 0$

On rearranging we get

$$\frac{dy}{dx} = -\frac{(xy + y^2)}{x^2}$$

$$\text{Let } f(x,y) = -\frac{(xy + y^2)}{x^2}$$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{(kxky + k^2y^2)}{k^2x^2}$$

Taking k^2 common we get

$$= \frac{k^2}{k^2} . -\frac{(xy + y^2)}{x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$x^2dy + (x y + y^2) dx = 0$

Above equation can be written as

$$\frac{dy}{dx} = -\frac{(xy + y^2)}{x^2}$$

To solve it we make the substitution.

$y = v x$

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting dy/dx value we get

$$v + x\frac{dv}{dx} = -\frac{(x.vx + v^2x^2)}{x^2}$$

$$v + x\frac{dv}{dx} = -\frac{(vx^2 + v^2x^2)}{x^2}$$

On computing and simplifying

$$v + x\frac{dv}{dx} = -v - v^2$$

$$x\frac{dv}{dx} = -v - v^2 - v$$

$$x\frac{dv}{dx} = -v(v + 2)$$

$$\frac{1}{v(v+2)}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Dividing and multiplying above equation by 2 we get

$$\frac{1}{2}\int \frac{2}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Adding and subtracting v to the numerator we get

$$\frac{1}{2}\int \frac{2+v-v}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Now splitting the denominator we get

$$\frac{1}{2}\int \left(\frac{2+v}{v(v+2)} - \frac{v}{v(v+2)}\right)dv = -\int \frac{1}{x}dx$$

$$\frac{1}{2}\int \left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\int \frac{1}{x}dx$$

On integrating we get

$$\frac{1}{2}(\log v - \log(v + 2)) = -\log x + \log C$$

Using logarithmic formula,

$$\frac{1}{2}\left(\log\frac{v}{v+2}\right) = \log\frac{C}{x}$$

$$\log\left(\frac{\frac{y}{x}}{\frac{y}{x}+2}\right) = 2\log\frac{C}{x}$$

$$\log\left(\frac{y}{y+2x}\right) = \log\left(\frac{C}{x}\right)^2$$

On simplification we get

$$\frac{y}{y+2x} = \left(\frac{C}{x}\right)^2$$

$$\frac{x^2y}{y+2x} = C^2$$

y = 1 when x = 1

$$C^2 = \frac{1}{1+2} = \frac{1}{3}$$

$$\therefore \frac{x^2y}{y+2x} = \frac{1}{3}$$

$3x^2y = y + 2x$

$y + 2x = 3x^2y$

The required solution of the differential equation.

13. $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0;\ y = \frac{\pi}{4}$ when $x = 1$

Solution:

Given

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx = -xdy$$

The above equation can be written as

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] = -x\frac{dy}{dx}$$

On rearranging

$$\frac{dy}{dx} = -\frac{\left[x \sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

We know f(x, y) = dy/dx using this in above equation we get

$$f(x,y) = -\frac{\left[x \sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{\left[kx \sin^2\left(\frac{ky}{kx}\right) - ky\right]}{kx}$$

Taking k as common

$$= \frac{k}{k}.-\frac{\left[x \sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + xdy = 0$$

On rearranging

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx = -xdy$$

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right] = -x\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{\left[x\sin^2\left(\frac{vx}{x}\right) - vx\right]}{x}$$

$$v + x\frac{dv}{dx} = -\left[\frac{x\sin^2 v - vx}{x}\right]$$

$$v + x\frac{dv}{dx} = -\sin^2 v - v$$

On computing and simplifying we get

$$x\frac{dv}{dx} = -[\sin^2 v - v] - v$$

$$x\frac{dv}{dx} = -\sin^2 v + v - v$$

$$x\frac{dv}{dx} = -\sin^2 v$$

$$\frac{1}{\sin^2 v}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int\frac{1}{\sin^2 v}dv = -\int\frac{1}{x}dx$$

$\int cosec^2 v dv$ = - log x – log C

On integrating we get

-cot v = - log x – log C

Cot v = log x + log C

Substituting the value of v we get

$\cot\frac{y}{x} = \log(Cx)$

$y = \frac{\pi}{4}$ when $x = 1$

$\cot\frac{\pi/4}{1} = \log(C.1)$

$\cot\frac{\pi}{4} = \log C$

1 = C

$e^1 = C$

$\therefore \cot\frac{y}{x} = \log(ex)$

The required solution of the differential equation.

14. $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0;\ y = 0$ when $x = 1$

Solution:
Given

$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$

On rearranging we get

$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

Let $f(x, y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

Now put x= k x and y = k y

$f(kx, ky) = \frac{ky}{kx} - \operatorname{cosec}\left(\frac{ky}{kx}\right)$

$= \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$

$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\left(\frac{vx}{x}\right)$$

On simplification

$$v + x\frac{dv}{dx} = v - \operatorname{cosec}v$$

$$x\frac{dv}{dx} = -\operatorname{cosec}v$$

$$\frac{1}{\operatorname{cosec}v}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \sin v\, dv = -\int \frac{1}{x}dx$$

On integrating we get

- Cos v = - log x + C

Substituting the value of v

$$-\cos\frac{y}{x} = -\log x + C$$

y = 0 when x = 1

$$-\cos\frac{0}{1} = -\log 1 + C$$

- 1 = C

$$\therefore -\cos\frac{y}{x} = -\log x - 1$$

$$\cos\frac{y}{x} = \log x + \log e$$

$$\cos\frac{y}{x} = \log|ex|$$

The required solution of the differential equation.

15. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0;\ \ y = 2 \text{ when } x = 1$

Solution:

Given

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

The above equation can be written as

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Let $f(x,y) = \dfrac{2xy + y^2}{2x^2}$

Now put x = k x and y = k y

$$f(kx, ky) = \frac{2kxky + (ky)^2}{2(kx)^2}$$

Taking k^2 common

$$= \frac{k^2}{k^2}\cdot\frac{2xy + y^2}{2x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

On rearranging

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{2x.vx + (vx)^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{2vx^2 + v^2x^2}{2x^2}$$

On computing and simplification we get

$$v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$$
$$x\frac{dv}{dx} = \frac{1}{2}v^2$$
$$2\frac{1}{v^2}dv = \frac{1}{x}dx$$
Taking integration on both sides, we get
$$\int 2\frac{1}{v^2}dv = \int \frac{1}{x}dx$$
On integrating we get
$$-\frac{2}{v} = \log x + C$$
Substituting the value of v we get
$$-\frac{2}{y/x} = \log x + C$$
$$-\frac{2x}{y} = \log x + C$$
y = 2 when x = 1
$$-\frac{2.1}{2} = \log 1 + C$$
- 1 = C
$$\therefore -\frac{2x}{y} = \log x - 1$$
$$\frac{2x}{y} = 1 - \log x$$
$$y = \frac{2x}{1 - \log|x|} : x \neq e, x \neq 0$$
The required solution of the differential equation.

16. A homogeneous differential equation of the from $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.
(A) y = v x **(B) v = y x** **(C) x = v y** **(D) x = v**

Solution:
(C) x = v y

Explanation:

Since, $\frac{dx}{dy}$ is given equal to $h\left(\frac{x}{y}\right)$.

Therefore,

$h\left(\frac{x}{y}\right)$ is a function of $\frac{x}{y}$.

Therefore, we shall substitute, x = v y is the answer

17. Which of the following is a homogeneous differential equation?

A. $(4x + 6y + 5)\, dy – (3y + 2x + 4)\, dx = 0$

B. $(x y)\, dx – (x^3 + y^3)\, dy = 0$

C. $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

D. $y^2dx + (x^2 – x y – y^2)\, dy = 0$

Solution:

D. $y^2dx + (x^2 – x y – y^2)\, dy = 0$

Explanation:

We have

$y^2dx + (x^2 – x y – y^2)\, dy = 0$

On rearranging

$$\frac{dy}{dx} = -\frac{x^2 - xy - y^2}{y^2}$$

$$\text{Let } f(x,y) = -\frac{x^2 - xy - y^2}{y^2}$$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{(kx)^2 - kxky - (ky)^2}{(ky)^2}$$

$$= \frac{k^2}{k^2} . -\frac{x^2 - xy - y^2}{y^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equations is homogeneous.

For each of the differential equations given in question, find the general solution:

1. $\dfrac{dy}{dx}+2y=\sin x$

Solution:

Given

$\frac{dy}{dx}+2y=\text{sinx}$

Given equation in the form of $\frac{dy}{dx}+py=Q$ where, p = 2 and Q = sin x

Now, I.F. = $e^{\int pdx}=e^{\int 2dx}=e^{2x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int(Q\times \text{I.F.})dx+C$

$\Rightarrow ye^{2x}=\int \text{sinx}.e^{2x}dx+C$........1

Let I = $\int \text{sinx}.e^{2x}dx$

Integrating using chain rule we get

$$\Rightarrow I=\ \text{sinx}\int e^{2x}dx-\int\left(\frac{d}{dx}(\text{sinx}).e^{\int 2dx}\right)dx$$

$$=\ \text{sinx}.\frac{e^{2x}}{2}-\int\left(\text{cosx}.\frac{e^{2x}}{2}\right)dx$$

On integrating and computing we get

$$=\frac{e^{2x}\text{sinx}}{2}-\frac{1}{2}\left[\text{cosx}\int e^{2x}-\int\left(\frac{d}{dx}(\text{cosx}).\int e^{2x}dx\right)dx\right]$$

$$=\frac{e^{2x}\text{sinx}}{2}-\frac{1}{2}\left[\text{cosx}\frac{e^{2x}}{2}-\int\left[(-\text{sinx}).\frac{e^{2x}}{2}\right]dx\right]$$

$$=\frac{e^{2x}\text{sinx}}{2}-\frac{e^{2x}\text{cosx}}{2}-\frac{1}{4}\int(\text{sinx}.e^{2x})dx$$

Above equation can be written as

$$=\frac{e^{2x}}{4}(2\text{sinx}-\text{cosx})-\frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^{2x}}{5}(2\sin x - \cos x)$$

Now, putting the value of I in 1, we get,

$$\Rightarrow ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

Therefore, the required general solution of the given differential equation is

$$y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

2. $\frac{dy}{dx} + 3y = e^{-2x}$

Solution:

Given

$\frac{dy}{dx} + 3y = e^{-2x}$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = 3 and Q = e^{-2x}

Now, I.F. = $e^{\int p dx} = e^{\int 3 dx} = e^{3x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.)dx + C$

$$\Rightarrow ye^{3x} = \int (e^{-2x} \times e^{2x})dx + C$$

$$\Rightarrow ye^{3x} = \int e^{x}dx + C$$

On integrating we get

$\Rightarrow ye^{3x} = e^{x} + C$

$\Rightarrow y = e^{-2x} + Ce^{-3x}$

Therefore, the required general solution of the given differential equation is y = $e^{-2x} + Ce^{-3x}$

3. $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution:

Given

$\frac{dy}{dx}+\frac{y}{x}=x^2$

This is equation in the form of $\frac{dy}{dx}+py=Q$

Where, $p=\frac{1}{x}$ and $Q = x^2$

Now, I.F. = $e^{\int pdx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int(Q\times \text{I.F.})dx+C$

$\Rightarrow y(x)=\int(x^2.x)dx+C$

$\Rightarrow xy=\int(x^3)dx+C$

On integrating we get

$\Rightarrow xy=\frac{x^4}{4}+C$

Therefore, the required general solution of the given differential equation is

$xy=\frac{x^4}{4}+C$

4. $\frac{dy}{dx}+(\sec x)y=\tan x\left(0\le x<\frac{\pi}{2}\right)$

Solution:

Given

$\frac{dy}{dx}+(\sec x)y=\tan x$

Given equation is in the form of $\frac{dy}{dx}+py=Q$

Where, p = sec x and Q = tan x)

Now, I.F. = $e^{\int pdx}=e^{\int \sec x dx}=e^{\log(\sec x+\tan x)}=\sec x+\tan x$

Thus, the solution of the given differential equation is given by the relation y (I.F.) = $\int(\mathrm{Q}\times\mathrm{I.F.})dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \sec x\tan x dx + \int \tan^2 x dx + C$

$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)dx + C$

$\Rightarrow$ y (sec x + tan x) = sec x + tan x – x+ C

Therefore, the required general solution of the given differential equation is

y (sec x + tan x) = sec x + tan x – x+ C.

$$5. \cos^2 x\frac{dy}{dx} + y = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$$

Solution:

Given

$\cos^2\frac{dy}{dx} + y = \tan x$

The above equation can be written as

$\Rightarrow \frac{dy}{dx} + \sec^2 x.y = \sec^2 x\tan x$

Given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\sec^2 x$ and Q = $\sec^2 x \tan x$

Now, I.F. = $e^{\int pdx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int(\mathrm{Q}\times\mathrm{I.F.})dx + C$

$\Rightarrow y.e^{\tan x} = \int e^{\tan x}dx + C$ 1

Now, Let t = tanx

$\Rightarrow \frac{d}{dx}(\tan x) = \frac{dt}{dx}$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow \sec^2 x dx = dt$

Thus, the equation 1 becomes,

$$\Rightarrow y.e^{\tan x} = \int (e^t.t)dt + C$$

$$\Rightarrow y.e^{\tan x} = \int (t.e^t)dt + C$$

Using chain rule for integration we get

$$\Rightarrow y.e^{\tan x} = t.\int e^t dt - \int \left(\frac{d}{dt}(t).\int e^t dt\right)dt + C$$

$$\Rightarrow y.e^{\tan x} = t.e^t - \int e^t dt + C$$

On integrating we get

$\Rightarrow te^{\tan x} = (t-1)e^t + C$

$\Rightarrow te^{\tan x} = (\tan x - 1)e^{\tan x} + C$

$\Rightarrow y = (\tan x - 1) + C\,e^{-\tan x}$

Therefore, the required general solution of the given differential equation is $y = (\tan x - 1) + C\,e^{-\tan x}$.

6. $x\dfrac{dy}{dx} + 2y = x^2 \log x$

Solution:

Given

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

The above equation can be written as

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, $p = \frac{2}{x}$ and $Q = x \log x$

Now, I.F. $= e^{\int p dx} = e^{\int \frac{2}{x}dx} = e^{2(\log x)} = e^{\log x^2} = x^2$

Thus, the solution of the given differential equation is given by the relation

$y\,(\text{I.F.}) = \int (Q \times \text{I.F.})dx + C$

$$\Rightarrow y.x^2 = \int (x\log x.x^2)dx + C$$

The above equation becomes

$$\Rightarrow x^2y = \int (x^3 \log x)dx + C$$

On integrating using chain rule we get

$$\Rightarrow x^2y = \log x.\int x^3 dx - \int \left[\frac{d}{dx}(\log x).\int x^3 dx\right] dx + C$$

$$\Rightarrow x^2y = \log x.\frac{x^4}{4} - \int \left(\frac{1}{x}.\frac{x^4}{4}\right) dx + C$$

$$\Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}\int x^3 dx + C$$

Integrating and simplifying we get

$$\Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}.\frac{x^4}{4} + C$$

$$\Rightarrow x^2y = \frac{1}{16}x^4(4\log x - 1) + C$$

$$\Rightarrow y = \frac{1}{16}x^2(4\log x - 1) + Cx^{-2}$$

Therefore, the required general solution of the given differential equation

$$y = \frac{1}{16}x^2(4\log x - 1) + Cx^{-2}$$

7. $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x}\log x$

Solution:

Given

$$x \log x \frac{dy}{dx} + y = \frac{2}{x}\log x$$

The above equation can be written as

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2}$$

The given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{1}{x \log x}$ and Q = $\frac{2}{x^2}$

Now, I.F. = $e^{\int p dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

Thus, the solution of the given differential equation is given by the relation:

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow y.\log x = \int \left[\frac{2}{x^2}.\log x\right]dx + C$ 1

Now, $\int \left[\frac{2}{x^2}.\log x\right]dx = 2\int \left(\log x.\frac{1}{x^2}\right)dx$

On integrating using chain rule we get

$= 2\left[\log x.\int \frac{1}{x^2}dx - \int \left\{\frac{d}{dx}(\log x).\int \frac{1}{x^2}dx\right\}dx\right]$

$= 2\left[\log x\left(-\frac{1}{x}\right) - \int \left(\frac{1}{x}.\left(-\frac{1}{x}\right)\right)dx\right]$

$= 2\left[-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right]$

$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$

$= -\frac{2}{x}(1 + \log x)$

Now, substituting the value in 1, we get,

$\Rightarrow y.\log x = -\frac{2}{x}(1 + \log x) + C$

Therefore, the required general solution of the given differential equation is

$y.\log x = -\frac{2}{x}(1 + \log x) + C$

8. $(1 + x^2)$ dy + 2xy dx = cot x dx $(x \neq 0)$

Solution:

Given

$(1 + x^2)dy + 2xydx = \cot x dx$

The above equation can be written as

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{(1 + x^2)} = \frac{\cot x}{1 + x^2}$

The given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{2x}{(1+x^2)}$ and Q = $\frac{\cot x}{1+x^2}$)

Now, I.F. = $e^{\int pdx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = 1 + x^2$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.)dx + C$

$$\Rightarrow y.(1 + x^2) = \int \left[\frac{\cot x}{1 + x^2}.(1 + x^2)\right]dx + C$$

$$\Rightarrow y.(1 + x^2) = \int \cot x dx + C$$

On integrating we get

$\Rightarrow y(1 + x^2) = \log|\sin x| + C$

Therefore, the required general solution of the given differential equation is

$y(1 + x^2) = \log|\sin x| + C$

9. $x\dfrac{dy}{dx} + y - x + xy\cot x = 0 \quad (x \neq 0)$

Solution:

Given

$x\frac{dy}{dx} + y - x + xy\cot x = 0$

The above equation can be written as

$$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1$$

The given equation is in the form of $\frac{dy}{dx} + px = Q$

Where, p = $\frac{1}{x} + \cot x$ and Q = 1

Now, I.F. = $e^{\int pdx} = e^{\int \left(\frac{1}{x} + \cot x\right)dy} = e^{\log x + \log(\sin x)} = e^{\log(x\sin x)} = x\sin x$

Thus, the solution of the given differential equation is given by the relation

x (I.F.) = $\int (Q \times I.F.)dy + C$

$$\Rightarrow y(x\sin x) = \int [1 \times x\sin x]dx + C$$

$$\Rightarrow y(x\sin x) = \int [x\sin x]dx + C$$

By splitting the integrals we get

$\Rightarrow y(x\sin x) = x\int \sin x dx - \int \left[\frac{d}{dx}(x).\int \sin x dx\right] + C$

$\Rightarrow y(x\sin x) = x(-\cos x) - \int 1.(-\cos x)dx + C$

On integrating we get

$\Rightarrow$ y (x sin x) = -x cos x + sin x + C

$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$

$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$

Therefore, the required general solution of the given differential equation is

$y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$

10. $(x+y)\frac{dy}{dx} = 1$

Solution:

Given

$(x+y)\frac{dy}{dx} = 1$

The above equation can be written as

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$

$\Rightarrow \frac{dx}{dy} = x + y$

$\Rightarrow \frac{dx}{dy} - x = y$

The given equation is in the form of $\frac{dy}{dx} + px = Q$

Where, p = -1 and Q = y

Now, I.F. = $e^{\int pdy} = e^{\int -dy} = e^{-y}$

Thus, the solution of the given differential equation is given by the relation:

x (I.F.) = $\int (Q \times I.F.)dy + C$

$\Rightarrow xe^{-y} = \int [y.e^{-y}]dy + C$

$$\Rightarrow xe^{-y} = y\int e^{-dy} - \int\left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy + C$$

$$\Rightarrow xe^{-y} = y(-e^{-y}) - \int(-e^{-y})dy + C$$

On integrating and computing we get

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

=> $x = - y - 1 + C\,e^{y}$

=> $x + y + 1 = C\,e^{y}$

Therefore, the required general solution of the given differential equation is $x + y + 1 = C\,e^{y}$.

11. y dx + (x – y²)dy = 0

Solution:

Given

$$ydx + (x - y^2)dy = 0$$

The above equation can be written as

$$\Rightarrow ydx = (y^2 - x)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{(y^2 - x)}{y} = y - \frac{x}{y}$$

On simplifying we get

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$$

The above equation is in the form of $\frac{dy}{dx} + px = Q$

Where, $p = \frac{1}{y}$ and $Q = y$

Now, I.F. = $e^{\int pdy} = e^{\int\frac{dy}{y}} = e^{\log y} = y$

Thus, the solution of the given differential equation is given by the relation

x (I.F.) = $\int(Q \times \text{I.F.})dy + C$

$$\Rightarrow x.y = \int[y.y]dy + C$$

$$\Rightarrow x.y = \int y^2 dy + C$$

On integrating we get

$$\Rightarrow x.y = \frac{y^3}{3} + C$$

$$\Rightarrow xy = \frac{y^3}{3} + \frac{C}{y}$$

Therefore, the required general solution of the given differential equation is

$xy = \frac{y^3}{3} + \frac{C}{y}$.

12. $(x+3y^2)\frac{dy}{dx} = y \ \ (y>0)$

Solution:

Given

$(x+3y^2)\frac{dy}{dx} = y$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x+3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x+3y^2}{y} = \frac{x}{y} + 3y$$

On simplification

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1/y and Q = 3y

Now, I.F. = $e^{\int p dy} = e^{-\int \frac{dy}{y}} = e^{-\log y} = e^{\log(\frac{1}{y})} = \frac{1}{y}$

Thus, the solution of the given differential equation is given by the relation:

x (I.F.) = $\int (Q \times I.F.) dy + C$

$$\Rightarrow x.\frac{1}{y} = \int \left[3y.\frac{1}{y}\right] dy + C$$

On integrating we get

$$\Rightarrow \frac{x}{y} = 3y + C$$

$\Rightarrow x = 3y^2 + Cy$

Therefore, the required general solution of the given differential equation is $x = 3y^2 + Cy$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx} + 2y\tan x = \sin x;\ y = 0 \text{ when } x = \frac{\pi}{3}$

Solution:

Given

$\frac{dy}{dx} + 2y\tan x = \sin x$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = 2 tanx and Q =sin x

Now, I.F. = $e^{\int p dx} = e^{\int 2\tan x dx} = e^{2\log(\sec x)} = e^{\log(\sec^2 x)} = \sec^2 x$

Thus, the solution of the given differential equation is given by the relation:

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow y.(\sec^2 x) = \int [\sin x.\sec^2 x]dx + C$

$\Rightarrow y.(\sec^2 x) = \int [\sec x.\tan x]dx + C$

On integrating we get

$\Rightarrow y.(\sec^2 x) = \sec x + C$ 1

Now, it is given that y = 0 at x = $\frac{\pi}{3}$

$0 \times \sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$

$\Rightarrow 0 = 2 + C$

$\Rightarrow C = -2$

Now, Substituting the value of C = -2 in 1, we get,

$\Rightarrow y.(\sec^2 x) = \sec x - 2$

$\Rightarrow y = \cos x - 2\cos^2 x$

Therefore, the required general solution of the given differential equation is
$y = \cos x - 2\cos^2 x$.

14. $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0$ when $x=1$

Solution:

Given

$(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx}+\frac{2xy}{(1+x^2)}=\frac{1}{(1+x^2)^2}$

The given equation is in the form of $\frac{dy}{dx}+py=Q$

Where, p = $\frac{2x}{(1+x^2)}$ and Q = $\frac{1}{(1+x^2)^2}$

Now, I.F. = $e^{\int pdx}=e^{\int\frac{2x}{(1+x^2)}dx}=e^{\log(1+x^2)}=1+x^2$

Thus, the solution of the given differential equation is given by the relation

y(I.F.) = $\int(Q\times I.F.)dx+C$

$\Rightarrow y.(1+x^2)=\int\left[\frac{1}{(1+x^2)^2}.(1+x^2)\right]dx+C$

$\Rightarrow y.(1+x^2)=\int\frac{1}{(1+x^2)}dx+C$

On integrating we get

$\Rightarrow y.(1+x^2)=\tan^{-1}x+C$ 1

Now, it is given that y = 0 at x = 1

0 = $\tan^{-1}1$+ C

$\Rightarrow$ C = $-\frac{\pi}{4}$

Now, Substituting the value of C = $-\frac{\pi}{4}$ in (1), we get,

$\Rightarrow y.(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$

Therefore, the required general solution of the given differential equation is

$y.(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$

15. $\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2$ when $x=\frac{\pi}{2}$

Solution:

Given

$\frac{dy}{dx} - 3y\cot x = \sin 2x$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -3cot x and Q = sin 2x

Now, I.F. = $e^{\int p dx} = e^{-3\int \cot x dx} = e^{-3\log|\sin x|} = e^{\log\left|\frac{1}{\sin^3 x}\right|} = \frac{1}{\sin^3 x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.)dx + C$

$\Rightarrow y.\frac{1}{\sin^3 x} = \int \left[\sin 2x.\frac{1}{\sin^3 x}\right] dx + C$

$\Rightarrow y\operatorname{cosec}^3 x = 2\int (\cot x \operatorname{cosec} x) dx + C$

On integrating we get

$\Rightarrow$ y $\operatorname{cosec}^3 x$ = 2cosecx + C

$\Rightarrow$ y = $-\frac{2}{\operatorname{cosec}^2 x} + \frac{3}{\operatorname{cosec}^3 x}$

$\Rightarrow$ y = $-2\sin^2 x + C\sin^3 x$............1

Now, it is given that y = 2 when x = $\frac{\pi}{2}$

Thus, we get,

2 = -2 + C

$\Rightarrow$ C = 4

Now, Substituting the value of C = 4 in 1, we get,

y = $-2\sin^2 x + 4\sin^3 x$

$\Rightarrow$ y = $4\sin^3 x - 2\sin^2 x$

Therefore, the required general solution of the given differential equation is

y = $4\sin^3 x - 2\sin^2 x$.

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution:

Let F(x, y) be the curve passing through origin and let (x, y) be a point on the curve.

We know the slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$

According to the given conditions, we get,

$$\frac{dy}{dx} = x + y$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} - y = x$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1 and Q = x

Now, I.F. = $e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I.F.}) dx + C$

$\Rightarrow ye^{-x} = \int xe^{-x} dx + C$ 1

Now, $\int xe^{-x} dx = x \int e^{-x} dx - \int \left[\frac{d}{dx}(x). \int e^{-x} dx\right] dx$

On integrating

$$= x(e^{-x}) - \int (-e^{-x}) dx$$

$$= x(e^{-x}) + (-e^{-x})$$

$$= -e^{-x}(x + 1)$$

Thus, from equation 1, we get,

$\Rightarrow ye^{-x} = -e^{-x}(x + 1) + C$

$\Rightarrow$ y = -(x+1) + C e^x

$\Rightarrow$ x + y + 1 = C e^x 2

Now, it is given that curve passes through origin.

Thus, equation 2 becomes

1 = C

$\Rightarrow$ C = 1

Substituting C = 1 in equation 2, we get,

x + y – 1 = e^x

Therefore, the required general solution of the given differential equation is

x + y -1 = e^x

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Solution:

Let F(x, y) be the curve and let (x, y) be a point on the curve.

We know the slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$

According to the given conditions, we get,

$$\frac{dy}{dx} + 5 = x + y$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} - y = x - 5$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1 and Q = x - 5

Now, I.F. = $e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I.F.}) dx + C$

$\Rightarrow ye^{-x} = \int (x-5)e^{-x} dx + C$ 1

Now, $\int (x-5)e^{-x} dx = (x-5)\int e^{-x} dx - \int \left[\frac{d}{dx}(x-5). \int e^{-x}\, dx\right] dx$

$= (x-5)(e^{-x}) - \int (-e^{-x}) dx$

On integrating we get

$= (x-5)(e^{-x}) + (-e^{-x})$

$= (4-x)e^{-x}$

Thus, from equation 1, we get,

$\Rightarrow ye^{-x} = (4-x)e^{-x} + C$

$\Rightarrow y = 4 - x + Ce^{x}$

$\Rightarrow x + y - 4 = Ce^{x}$

Thus, equation (2) becomes:

$0 + 2 - 4 = C\, e^{0}$

$\Rightarrow -2 = C$

$\Rightarrow C = -2$

Substituting C = -2 in equation (2), we get,
$x + y - 4 = -2e^x$
$\Rightarrow y = 4 - x - 2e^x$
Therefore, the required general solution of the given differential equation is
$y = 4 - x - 2e^x$

18. The Integrating Factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ **is**

A. e^{-x} **B.** e^{-y} **C. 1/x** **D. x**

Solution:
C. 1/x

Explanation:
Given
$x\frac{dy}{dx} - y = 2x^2$
On simplification we get
$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x$
This is equation in the form of $\frac{dy}{dx} + py = Q$
Where, p = -1/x and Q =2x
Now, I.F. = $e^{\int p dx} = e^{\int -\frac{1}{x} dx} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$
Hence the answer is 1/x

19. The Integrating Factor of the differential equation
$(1-y^2)\frac{dx}{dy} + yx = ay(-1 < y < 1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Solution:
(D) $\frac{1}{\sqrt{1-y^2}}$

Explanation:

Given

$(1-y^2)\frac{dy}{dx}+yx=ay$

On rearranging we get

$\Rightarrow \frac{dy}{dx}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$

This is equation in the form of $\frac{dy}{dx}+py=Q$

Where, p = $\frac{y}{1-y^2}$ and Q = $\frac{a}{1-y^2}$

Now, I.F. =

$$e^{\int p dy}=e^{\int \frac{y}{1-y^2}dy}=e^{\frac{1}{2}\log(1-y^2)}=e^{\log\left[\frac{1}{\sqrt{(1-y^2)}}\right]}$$

$$=\frac{1}{\sqrt{(1-y^2)}}$$

MISCELLANEOUS EXERCISE

1. For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

(iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution:

(i) Given

$\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

On rearranging we get

$\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x=0$

We can see that the highest order derivative present in the differential is $\frac{d^2y}{dx^2}$

Thus, its order is two. It is polynomial equation in $\frac{d^2y}{dx^2}$

The highest power raised to $\frac{d^2y}{dx^2}$ is 1.

Therefore, its degree is one.

(ii) Given

$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

The above equation can be written as

$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$

We can see that the highest order derivative present in the differential is $\frac{dy}{dx}$

Thus, its order is one. It is polynomial equation in $\frac{dy}{dx}$

The highest power raised to $\frac{dy}{dx}$ is 3.

Therefore, its degree is three.

(iii) Given

$$\frac{d^4y}{dx^4} - \sin\left(\frac{dy}{dx}\right)^3 = 0$$

The above equation can be written as

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y - \log x = 0$$

We can see that the highest order derivative present in the differential is $\frac{d^4y}{dx^4}$
Thus, its order is four. The given differential equation is not a polynomial equation.
Therefore, its degree is not defined.

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $xy = a\,e^x + b\,e^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^x\,(a\cos x + b\sin x)$: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(iii) $y = x\sin 3x$: $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2\log y$: $(x^2 + y^2)\frac{dy}{dx} - xy = 0$

Solution:

(i) Given $x\,y = a\,e^x + b\,e^{-x} + x^2$

Now, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = a\frac{d}{dx}(e^x) + b\frac{d}{dx}(e^{-x}) + \frac{d}{dx}(x^2)$$

$$\Rightarrow \frac{dy}{dx} = ae^x - be^{-x} + 2x$$

Now, again differentiating above equation both sides with respect to x, we get,

$$\frac{d}{dx}(y') = \frac{d}{dx}(ae^x - be^{-x} + 2x)$$
$$\Rightarrow \frac{d^2y}{dx^2} = ae^x + be^{-x} + 2$$

Now, Substituting the values of $\frac{dy}{dx}$' and $\frac{d^2y}{dx^2}$ in the given differential equations, we get,

We have

LHS = $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$

$= x\,(ae^x + be^{-x} + 2) + 2(ae^x - be^{-x} + 2) - x\,(ae^x + be^{-x} + x^2) + x^2 - 2$

$= (axe^x + bxe^{-x} + 2x) + 2(ae^x - be^{-x} + 2) - x\,(ae^x + be^{-x} + x^2) + x^2 - 2$

$= 2ae^x - 2be^{-x} + x^2 + 6x - 2$

$\neq 0$

$\Rightarrow$ LHS $\neq$ RHS.

Therefore, the given function is not the solution of the corresponding differential equation.

(ii) Given $y = e^x(a \cos x + b \sin x) = ae^x \cos x + b\, e^x \sin x$

Now, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = a\frac{d}{dx}(e^x\cos x) + b\frac{d}{dx}(e^x\sin x)$$
$$\Rightarrow \frac{dy}{dx} = a(e^x\cos x - e^x\sin x) + b.(e^x\sin x + e^x\cos x)$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = (a + b)e^x\cos x + (b - a)e^x\sin x$$

Now, again differentiating both sides with respect to x, we get,

$$\frac{d^2y}{dx^2} = (a + b).\frac{d}{dx}(e^x\cos x) + (b - a)\frac{d}{dx}(e^x\sin x)$$

Taking common

$$= (a + b).[e^x\cos x - e^x\sin x] + (b - a)[e^x\sin x + e^x\cos x]$$

Simplifying we get

$$= e^x[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x]$$
$$= [2e^x(b\cos x - a\sin x)]$$

Now, Substituting the values of $\frac{dy}{dx}$' and $\frac{d^2y}{dx^2}$ in the given differential equations, we get,

LHS = $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y$

$=2e^x(b\cos x - a\sin x) - 2e^x[(a+b)\cos x + (b-a)\sin x] + 2e^x(a\cos x + b\sin x)$

$=e^x[(2b\cos x - 2a\sin x) - (2a\cos x + 2b\cos x) - (2b\sin x - 2a\sin x) + (2a\cos x + 2b\sin x)]$

$= e^x[(2b - 2a - 2b + 2a)\cos x] + e^x[(-2a - 2b + 2a + 2b\sin x]$

= 0 = RHS.

Therefore, the given function is the solution of the corresponding differential equation.

(iii) It is given that y = xsin3x

Now, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x\sin 3x) = \sin 3x + x.\cos 3x.3$$

$$\Rightarrow \frac{dy}{dx} = \sin 3x + 3x\cos 3x$$

Now, again differentiating both sides with respect to x, we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\sin 3x) + 3\frac{d}{dx}(x\cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\text{xos}3x + 3[\cos 3x + x(-\sin 3x).3]$$

On simplifying we get

$$\Rightarrow \frac{d^2y}{dx^2} = 6\cos 3x - 9x\sin 3x$$

Now, substituting the value of $\frac{d^2y}{dx^2}$ in the LHS of the given differential equation, we get,

$$\frac{d^2y}{dx^2} + 9y - 6\cos 3x$$

= (6.cos3x – 9xsin3x) + 9xsin3x – 6cos3x

= 0 = RHS

Therefore, the given function is the solution of the corresponding differential equation.

(iv) Given $x^2 = 2y^2 \log y$

Now, differentiating both sides with respect to x, we get,

$2x = 2.\frac{d}{dx}(y^2 \log y)$

Using product rule we get

$$\Rightarrow x = \left[2y.\log y.\frac{dy}{dx} + y^2.\frac{1}{y}.\frac{dy}{dx}\right]$$

$$\Rightarrow x = \frac{dy}{dx}(2y\log y + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1 + 2\log y)}$$

Now, substituting the value of $\frac{dy}{dx}$ in the LHS of the given differential equation, we get,

$$(x^2 + y^2)\frac{dy}{dx} - xy = (2y^2\log y + y^2).\frac{x}{y(1 + 2\log y)} - xy$$

$$= y^2(1 + 2\log y).\frac{x}{y(1 + 2\log y)} - xy$$

$= xy - xy$

$= 0$

Therefore, the given function is the solution of the corresponding differential equation.

3. Form the differential equation representing the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Solution:

Given $(x - a)^2 + 2y^2 = a^2$

$\Rightarrow x^2 + a^2 - 2ax + 2y^2 = a^2$

$\Rightarrow 2y^2 = 2ax - x^2$..........1

Now, differentiating both sides with respect to x, we get,

$$2y\frac{dy}{dx} = \frac{2a - 2x}{2}$$

On simplifying we get

$$\Rightarrow \frac{dy}{dx} = \frac{a - x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ax-2x^2}{4xy} \quad ....2$$

So, equation (1), we get,

$2ax = 2y^2 + x^2$

On substituting this value in equation 2, we get,

$$\frac{dy}{dx} = \frac{2y^2 + x^2 - 2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$$

Therefore, the differential equation of the family of curves is given as

$$\frac{dy}{dx} = \frac{2y^2-x^2}{4xy}$$

4. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

Solution:

Given $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y} \quad ......1$$

Now, let us take y = vx for further simplification

On differentiating we get

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, substituting the values of y and dv/dx in equation 1, we get,

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$$

Taking common and simplifying we get

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

Taking LCM and simplifying we get

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2-v(v^3-3v)}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^4}{v^3-3v}$$

$$\Rightarrow \left(\frac{v^3-3v}{1-3v^4}\right)dv = \frac{dx}{x}$$

On integrating both sides we get,

$$\int\left(\frac{v^3-3v}{1-3v^4}\right)dv = \log x + \log C' \quad ....2$$

Splitting the denominator

Now, $\int\left(\frac{v^3-3v}{1-3v^4}\right)dv = \int\frac{v^3}{1-v^4}dv - 3\int\frac{vdv}{1-v^4}$

$$\Rightarrow \int\left(\frac{v^3-3v}{1-3v^4}\right)dv = I_1 - 3I_2, \text{where } I_1 = \int\frac{v^3}{1-v^4}dv \text{ and } I_2 = \int\frac{vdv}{1-v^4} \quad ....3$$

Let $1 - v^4 = t$

On differentiating we get

$$\Rightarrow \frac{d}{dv}(1-v^4) = \frac{dt}{dv}$$

$$\Rightarrow -4v^3 = \frac{dt}{dv}$$

$$\Rightarrow v^3dv = -\frac{dt}{4}$$

Now, $I_1 = \int -\frac{dt}{4} = -\frac{1}{4}\log t = -\frac{1}{4}\log(1-v^4)$

and $I_2 = \int\frac{vdv}{1-v^4} = \int\frac{vdv}{1-(v^2)^2}$

Let $v^2 = p$

Differentiating above equation with respect to v

$$\Rightarrow \frac{d}{dv}(v^2) = \frac{dp}{dv}$$

$$\Rightarrow 2v = \frac{dp}{dv}$$

$$\Rightarrow vdv = \frac{dp}{2}$$

Using these things we get

$$\therefore I_2 = \frac{1}{2}\int \frac{dp}{1-p^2} = \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right| = \frac{1}{4}\left|\frac{1+v^2}{1-v}\right|$$

Now, substituting the values of I_1 and I_2 in equation (3), we get,

$$\int\left(\frac{v^3-3y}{1-v^4}\right)dv = -\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Thus, equation (2), becomes,

$$-\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$$

Computing and simplifying we get

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$$

$\Rightarrow (x^2 - y^2)^2 = C'^4(x^2 + y^2)^4$

$\Rightarrow (x^2 - y^2) = C'^2(x^2 + y^2)$

$\Rightarrow (x^2 - y^2) = C(x^2 + y^2)$, where $C = C'^2$

Therefore, the result is proved.

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Solution:

We know that the equation of a circle in the first quadrant with centre (a, a) and radius a which touches the coordinate axes is $(x - a)^2 + (y - a)^2 = a^2$ …………1

Now differentiating above equation with respect to x, we get,

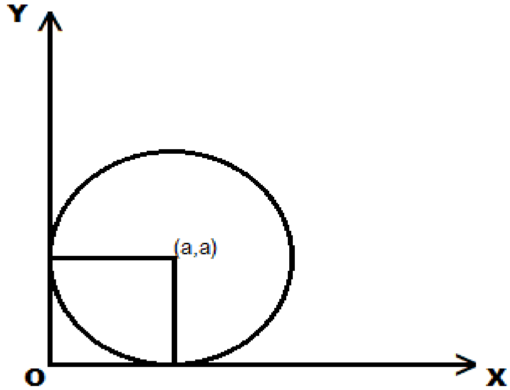
$2(x-a) + 2(y-a)\, dy/dx = 0$

$\Rightarrow (x - a) + (y - a)\, y' = 0$

On multiplying we get

$\Rightarrow x - a + yy' - ay' = 0$

$\Rightarrow x + yy' - a(1+y') = 0$



Therefore from above equation we have

$\Rightarrow a = \frac{x+yy'}{1+y'}$

Now, substituting the value of a in equation 1, we get,

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

Taking LCM and simplifying we get

$$\Rightarrow\left[\frac{(x-y)y'}{1+y'}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$\Rightarrow (x - y)^2.y'^2 + (x - y)^2 = (x + yy')^2$

$\Rightarrow (x - y)^2[1 + (y')^2] = (x + yy')^2$

Therefore, the required differential equation of the family of circles is

$(x - y)^2[1 + (y')^2] = (x + yy')^2$

6. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Solution:

Given

$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

On rearranging we get

$$\Rightarrow\frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow\frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$$

On integrating, we get,
$\sin^{-1} y = \sin^{-1} x + C$
$\Rightarrow \sin^{-1}x + \sin^{-1}y = C$

7. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by (x + y + 1) = A (1 – x – y – 2xy), where A is parameter.

Solution:

Given
$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$
On rearranging
$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y^2+y+1}{x^2+x+1}\right)$$
Separating the variables using variable separable method we get
$$\Rightarrow \frac{dy}{y^2+y+1} = \frac{-dx}{x^2+x+1}$$
$$\Rightarrow \frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$$
Taking integrals on both sides, we get,
$$\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C$$
$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dy}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = C$$
On integrating we get
$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = C$$
$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + \tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = C$$
Using $\tan^{-1}$ formula we get

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{2y+1}{\sqrt{3}}.\frac{2x+1}{\sqrt{3}}}\right]=\frac{\sqrt{3}}{2}C$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right]=\frac{\sqrt{3}}{2}C$$

Computing and simplifying we get

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right]=\frac{\sqrt{3}}{2}C$$

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right]=\frac{\sqrt{3}}{2}C$$

$$\Rightarrow \frac{\sqrt{3}(x+y+1)}{(1-x-y-2xy)}=\tan\left(\frac{\sqrt{3}}{2}C\right)$$

Let $\tan\left(\frac{\sqrt{3}}{2}C\right)=B$

Then,

$$x+y+1=\frac{2B}{\sqrt{3}}(1-x-y-2xy)$$

Now, let A = $\frac{2B}{\sqrt{3}}$ is a parameter, then, we get

$$x+y+1=A(1-x-y-2xy)$$

8. Find the equation of the curve passing through the point (0, π/4) whose differential equation is sin x cos y dx + cos x sin y dy = 0.

Solution:

Given sin x cos y dx + cos x sin y dy = 0

Dividing the given equation by cos x cos y we get

$$\Rightarrow \frac{\sin x\cos y dx+\cos x\sin y dy}{\cos x\cos y}=0$$

On simplification we get

⇒ Tan x dx + tan y dy = 0

So, on integrating both sides, we get,

Log (sec x) + log (sec y) = log C

Using logarithmic formula we get

⇒ Log (sec x sec y) = log C

⇒ Sec x sec y = C

The curve passes through point (0, π/4)

Thus, 1 × √2 = C

⇒ C = √2

On substituting C = √2 in equation (1), we get,

Sec x sec y = √2

$$\Rightarrow \sec x.\frac{1}{\cos y} = \sqrt{2}$$

$$\Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Therefore, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$

9. Find the particular solution of the differential equation $(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$, given that y = 1 when x = 0.

Solution:

Given $(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$

Separating the variables using variable separable method we get

$$\Rightarrow \frac{dy}{1 + y^2} + \frac{e^x dx}{1 + e^{2x}} = 0$$

On integrating both sides, we get,

$$\tan^{-1} y + \int \frac{e^x dx}{1+e^{2x}} = C \quad \ldots\ldots 1$$

Let $e^x = t$

$\Rightarrow e^{2x} = t^2$

On differentiating we get

$$\Rightarrow \frac{d}{dx}(e^x) = \frac{dt}{dx}$$

$$\Rightarrow e^x = \frac{dt}{dx}$$

$\Rightarrow e^x dx = dt$

Substituting the value in equation (1), we get,

$$\tan^{-1} y + \int \frac{dt}{1+t^2} = C$$

$\Rightarrow \tan^{-1} y + \tan^{-1} t = C \Rightarrow \tan^{-1} y + \tan^{-1} (e^x) = C$ 2

Now, y =1 at x = 0

Therefore, equation (2) becomes,

$\tan^{-1} 1 + \tan^{-1} 1 = C$

$$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C$$

$$\Rightarrow C = \frac{\pi}{4}$$

Substituting c = π/4 in (2), we get,

$\tan^{-1} y + \tan^{-1} (e^x) = \frac{\pi}{4}$

10. Solve the differential equation $y e^{\frac{x}{y}} dx = \left(x e^{\frac{x}{y}} + y^2 \right) dy \ (y \neq 0)$

Solution:

Given

$$ye^{\frac{x}{y}} dx = \left(xe^{\frac{x}{y}} + y^2\right) dy$$

On rearranging we get

$$\Rightarrow ye^{\frac{x}{y}} \frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$$

Taking common

$$\Rightarrow e^{\frac{x}{y}} \left[y.\frac{dx}{dy} - x\right] = y^2$$

$$\Rightarrow e^{\frac{x}{y}}.\frac{\left[y.\frac{dx}{dy} - x\right]}{y^2} = 1$$ 1

Let $e^{\frac{x}{y}} = z$

Differentiating it with respect to y, we get,

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}.\frac{d}{dy}\left(\frac{x}{y}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}.\left[\frac{y.\frac{dx}{dy} - x}{y^2}\right] = \frac{dz}{dy}$$ 2

From equation (1) and equation (2), we have

$$\frac{dz}{dy} = 1$$

$\Rightarrow dz = dy$

On integrating both sides, we get,

$z = y + C$

$$\Longrightarrow e^{\frac{x}{y}} = y + C$$

11. Find a particular solution of the differential equation (x – y) (dx + dy) = dx – dy, given that y = –1, when x = 0. (Hint: put x – y = t)

Solution:

Given $(x - y)(dx + dy) = dx - dy$

$\Rightarrow (x - y + 1)\, dy = (1 - x + y)\, dx$

On rearranging we get

$$\Longrightarrow \frac{dy}{dx} = \frac{1 - x + y}{x - y + 1}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1-(x-y)}{1+(x+y)} \quad \text{............1}$$

Let $x - y = t$

Differentiating above equation with respect to x we get

$$\Longrightarrow \frac{d(x-y)}{dx} = \frac{dt}{dx}$$

$$\Longrightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Longrightarrow 1 - \frac{dt}{dx} = \frac{dy}{dx}$$

Now, let us substitute the value of x-y and $\frac{dy}{dx}$ in equation (1), we get,

$$1 - \frac{dt}{dx} = \frac{1-t}{1+t}$$

On rearranging we get

$$\Longrightarrow \frac{dt}{dx} = 1 - \left(\frac{1-t}{1+t}\right)$$

$$\Longrightarrow \frac{dt}{dx} = \frac{(1+t)-(1-t)}{1+t}$$

Computing and simplifying we get

$$\Rightarrow \frac{dt}{dx} = \frac{2t}{1+t}$$

$$\Rightarrow \left(\frac{1+t}{t}\right) dt = 2dx$$

$$\Rightarrow \left(1+\frac{1}{t}\right) dt = 2dx \quad \text{..........2}$$

On integrating both side, we get,

t + log |t| = 2x + C

⇒ (x – y) + log |x – y| = 2x + C

⇒ Log |x – y| = x + y + C 3

Now, y = -1 at x = 0

Then, equation (3), we get,

Log 1 = 0 -1 + C

⇒ C = 1

Substituting C = 1in equation (3), we get,

Log |x – y| = x + y + 1

Therefore, a particular solution of the given differential equation is log |x – y| = x + y + 1.

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1\ (x \neq 0)$

Solution:

Given

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dy}{dx} = 1$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{1}{\sqrt{x}}$ and Q = $\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now, I.F. = $e^{\int p dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.) dx + C$

$$\Rightarrow ye^{2\sqrt{x}} = \int \left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\right) dx + C$$

$$\Rightarrow ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + C$$

On integrating we get

$$\Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + C$$

13. Find a particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$ **(x ≠ 0), given that y = 0 when x = π/2**

Solution:

Given

$$\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$$

Given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = cot x and Q = 4x cosec x

Now, I.F. = $e^{\int p dx} = e^{\int \cot x dx} = e^{\log|\sin x|} = \sin x$

Thus, the solution of the given differential equation is given by the relation

y(I.F.) = $\int (Q \times I.F.) dx + C$

$$\Rightarrow y\sin x = \int 2x\,\mathrm{cosec}\,x\,dx + C$$

$$= 4\int x\,dx + C$$

On integrating we get

$$= 4.\frac{x^2}{2} + C$$

$$\Rightarrow y\sin x = 2x^2 + C \text{.........1}$$

Now, y = 0 at x = $\frac{\pi}{2}$

Therefore, equation (1), we get,

$0 = 2 \times \frac{\pi^2}{4} + C$

$\Rightarrow C = \frac{\pi^2}{4}$

Now, substituting $C = \frac{\pi^2}{4}$ in equation (1), we get,

$y \sin x = 2x^2 - \frac{\pi^2}{4}$

Therefore, the required particular solution of the given differential equation is

$y \sin x = 2x^2 - \frac{\pi^2}{4}$

14. Find a particular solution of the differential equation, $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$ given that y = 0 when x = 0.

Solution:

Given

$(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$

On rearranging we get

$\Rightarrow \frac{dy}{2e^{-y} - 1} = \frac{dx}{x + 1}$

$\Rightarrow \frac{e^y dy}{2 - e^y} = \frac{dx}{x + 1}$

On integrating both sides, we get,

$\int \frac{e^y dy}{2 - e^y} = \log|x + 1| + \log C$ 1

Let $2 - e^y = t$

$\therefore \frac{d}{dy}(2 - e^y) = \frac{dt}{dy}$

$\Rightarrow -e^y = \frac{dt}{dy}$

$\Rightarrow e^y dt = -dt$

Substituting value in equation (1), we get,

$\int \frac{-dt}{t} = \log|x + 1| + \log c$

On integrating we get

$\Rightarrow -\log |t| = \log |C(x+1)|$

$\Rightarrow -\log|2 - e^y| = \log |C(x + 1)|$

$\Rightarrow \frac{1}{2-e^y} = C(x+1)$

$\Rightarrow 2 - e^y = \frac{1}{c(x+1)}$............2

Now, at x = 0 and y = 0, equation (2) becomes,

$\Rightarrow 2 - 1 = \frac{1}{C}$

$\Rightarrow C = 1$

Now, substituting the value of C I equation (2), we get,

$\Rightarrow 2 - e^y = \frac{1}{(x+1)}$

$\Rightarrow e^y = 2 - \frac{1}{(x+1)}$

$\Rightarrow e^y = \frac{2x+2-1}{(x+1)}$

$\Rightarrow e^y = \frac{2x+1}{(x+1)}$

$\Rightarrow y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1)$

Therefore, the required particular solution of the given differential equation is

$y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1)$

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let the population at any instant (t) be y.

Now it is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$\therefore \frac{dy}{dt} \alpha y$

$\Rightarrow \frac{dy}{dt} = ky$

Where k is proportionality constant.

$\Rightarrow \frac{dy}{y} = kdt$

Now, integrating both sides, we get,

Log y = k t + C 1

According to given conditions,

In the year 1999, t = 0 and y = 20000

$\Rightarrow$ log20000 = C 2

Also, in the year 2004, t = 5 and y = 25000

$\Rightarrow$ Log 25000 = k.5 + C

$\Rightarrow$ log 25000 = 5k + log 20000

$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$ 3

Also, in the year 2009, t = 10

Now, substituting the values of t, k and c in equation (1), we get

Log y = $10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$

$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$

$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow$ y = 31250

Therefore, the population of the village in 2009 will be 31250.

16. The general solution of the differential equation $\frac{ydx - xdx}{y} = 0$ is

A. xy = C **B. x = Cy^2** **C. y = Cx** **D. y = Cx^2**

Solution:

C. y = Cx

Explanation:

Given question is

$\Rightarrow \frac{ydx - xdx}{xy} = 0$

On rearranging we get

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating both sides, we get,

Log |x| - log |y| = log k

$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$

$\Rightarrow \frac{x}{y} = k$

$\Rightarrow y = \frac{1}{k}x$

$\Rightarrow$ y = Cx where C = $\frac{1}{k}$

17. The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ **is**

(A) $y\, e^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy}\right) dy + C$

(B) $y \,.\, e^{\int P_1\, dx} = \int \left(Q_1 e^{\int P_1\, dx}\right) dx + C$

(C) $x\, e^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy}\right) dy + C$

(D) $x\, e^{\int P_1\, dx} = \int \left(Q_1 e^{\int P_1\, dx}\right) dx + C$

Solution:

(C) $x\, e^{\int P_1\, dy} = \int \left(Q_1 e^{\int P_1\, dy}\right) dy + C$

Explanation:

The integrating factor of the given differential equation $\frac{dx}{dy} + P_1 x = Q_1$ is $e^{\int P_1 dy}$.

Thus, the general solution of the differential equation is given by,

$$x(\text{I.F.}) = \int (Q \times \text{I.F.})dy + C$$
$$\Rightarrow x.\, e^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right)dy + C$$

18. The general solution of the differential equation e^x dy + (y e^x + 2x) dx = 0 is
A. x ey + x^2 = C B. x ey + y^2 = C C. y ex + x^2 = C D. y ey + x^2 = C

Solution:
C. y ex + x^2 = C

Explanation:
Given $e^x dy + (ye^x + 2x)\, dx = 0$
On rearranging we get
$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$
This is equation in the form of $\frac{dy}{dx} + py = Q$
Where, p = 1 and Q = $-2xe^{-x}$
Now, I.F. = $e^{\int p dx} = e^{\int dx} = e^x$
Thus, the solution of the given differential equation is given by the relation
y (I.F.) = $\int (Q \times \text{I.F.})dx + C$
$$\Rightarrow ye^x = \int (-2xe^{-x}.\, e^x)dx + C$$
$$\Rightarrow ye^x = -\int 2x dx + C$$
On integrating we get
$\Rightarrow ye^x = -x^2 + C$
$\Rightarrow ye^x + x^2 = C$