Exercise 8.1

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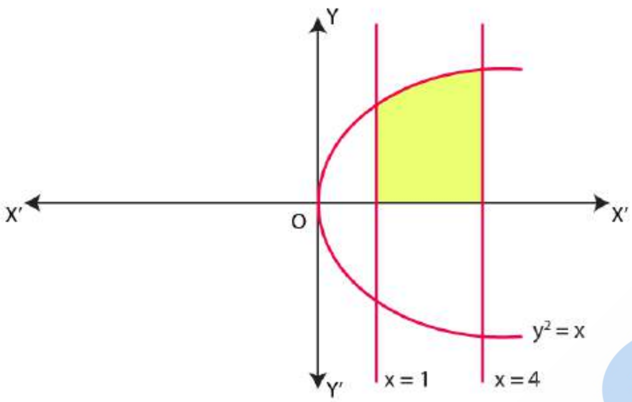
1. Find the area of the region bounded by the curve $y^2 = x$ and the lines x=1, x = 4 and the x- axis in the first quadrant.

Solution: Equation of the curve (rightward parabola) is $y^2 = x$.



$y = \sqrt{x}$ ....(1)

Required area is shaded region:

$= \left|\int_1^4 y\,dx\right| = \left|\int_1^4 \sqrt{x}\,dx\right|$ [From equation (1)]

$= \left|\int_1^4 x^{\frac{1}{2}}\,dx\right|$

$= \left|\frac{\left(x^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$

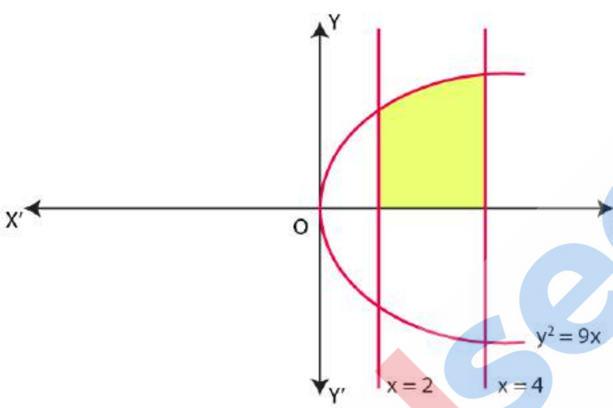
$= \left|\frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\right|$

$= \left|\frac{2}{3}\left(4^{\frac{1}{2}\times 3} - 1^{\frac{1}{2}\times 3}\right)\right| = \left|\frac{2}{3}(8-1)\right| = \frac{2}{3}\times 7 = \frac{14}{3}$ sq. units

2. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

Solution: Equation of the curve (rightward parabola) is $y^2 = 9x.$

$y = 3\sqrt{x}$ ……….(1)



Required area is shaded region, which is bounded by curve $y^2 = 9x,$ and vertical lines x=2, x=4 and x-axis in first quadrant.

$= \left|\int_2^4 y\, dx\right| = \left|\int_1^4 3\sqrt{x}\, dx\right|$ [From equation (1)]

$= \left|3\int_2^4 x^{\frac{1}{2}}\, dx\right| = \left|3\frac{\left(x^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right|$

$= \left|3.\frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)\right| = \left|3.\frac{2}{3}\left(4^{\frac{1}{2}\times 3} - 2^{\frac{1}{2}\times 3}\right)\right|$

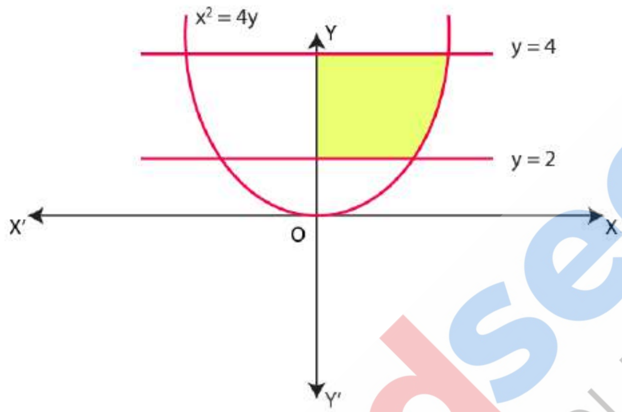
$= \left|2\left(8-2\sqrt{2}\right)\right| = \left(16-4\sqrt{2}\right)$ sq. units

3. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y- axis in the first quadrant.

Solution: Equation of curve (parabola) is $x^2 = 4y$.

or $x = 2\sqrt{y}$ ……….(1)

Required region is shaded, that is area bounded by curve $x^2 = 4y$, and Horizontal lines $y = 2, y = 4$ and y-axis in first quadrant.

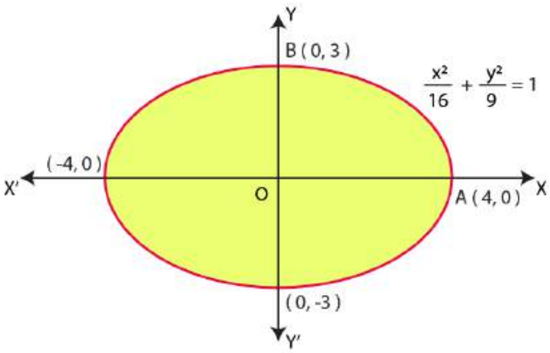


$= \left|\int_2^4 x\,dy\right| = \left|\int_2^4 2\sqrt{y}\,dy\right| = \left|2\int_2^4 y^{\frac{1}{2}}\,dy\right|$

$= \left|2\frac{\left(y^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right| = \frac{4}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \left(\frac{32-8\sqrt{2}}{3}\right)$ sq. units

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution: Equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ……….(1)



Here $a^2(=16) > b^2(=9)$

From equation (1), $\frac{y^2}{9} = 1 - \frac{x^2}{16} = \frac{16-x^2}{16}$

$\Rightarrow y^2 = \frac{9}{16}(16-x^2)$

$\Rightarrow y^2 = \frac{3}{4}(16-x^2)$ ……….(2)

for arc of ellipse in first quadrant.

Ellipse (1) is symmetrical about x-axis and about y-axis (if we change y to –y or x to –x, equation remain same).

Intersections of ellipse (1) with x-axis (y=0)
Put y=0 in equation (1), we have

$\frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Therefore, Intersections of ellipse (1) with x-axis are (0, 4) and (0, -4).

Now again,
Intersections of ellipse (1) with y-axis (x=0)

Putting x=0 in equation (1), $\frac{y^2}{9}=1 \Rightarrow y^2=9 \Rightarrow y=\pm 3$

Therefore, Intersections of ellipse (1) with y-axis are (0, 3) and (0,-3).

Now ,

Area of region bounded by ellipse (1) = Total shaded area = 4 x Area OAB of ellipse in first quadrant

= $4\left|\int_0^4 y\,dx\right|$ [∵ At end B of arc AB of ellipse; $x=0$ and at end A of arc AB ; $x=4$]

= $4\left|\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx\right|$ = $4\left|\int_0^4 \frac{3}{4}\sqrt{4^2-x^2}\,dx\right|$

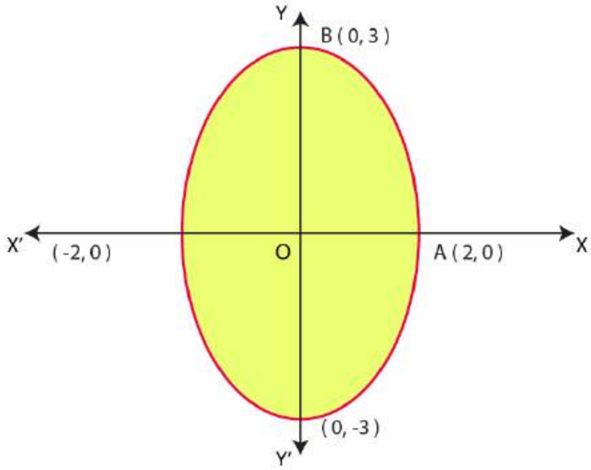
= $3\left[\frac{x}{2}\sqrt{4^2-x^2}+\frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$ $\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

= $3\left[\frac{4}{2}\sqrt{16-16}+8\sin^{-1}1-\left(0+8\sin^{-1}0\right)\right]$ = $3\left[0+\frac{8\pi}{2}\right]$

= $3(4\pi)=12\pi$ sq. units

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution: Equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{9}=1$



Here $a^2(=4)<b^2(=9)$

From equation (1), $\frac{y^2}{9}=1-\frac{x^2}{4}=\frac{4-x^2}{4}$

$\Rightarrow y^2=\frac{9}{4}(4-x^2)$

$\Rightarrow y^2=\frac{3}{2}(4-x^2)$ ……….(2)

For an arc of ellipse in first quadrant.

Ellipse (1) is symmetrical about x-axis and y-axis.

Intersections of ellipse (1) with x-axis (y=0)

Put y=0 in equation (1), $\frac{x^2}{4}=1$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

Therefore, Intersections of ellipse (1) with x-axis are (0, 2) and (0,-2).

Intersections of ellipse (1) with y-axis (x=0)

Put $x=0$ in equation (1), $\frac{y^2}{9}=1$

$\Rightarrow y^2=9 \Rightarrow y=\pm 3$

Therefore, Intersections of ellipse (1) with y-axis are (0, 3) and (0,-3).

Now,

Area of region bounded by ellipse (1) = Total shaded area = 4 x Area OAB of ellipse in first quadrant

= $4\left|\int_0^2 y\,dx\right|$ [∵ At end B of arc AB of ellipse; $x=0$ and at end A of arc AB ; $x=2$]

= $4\left|\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx\right|$ = $4\left|\int_0^4 \frac{3}{2}\sqrt{2^2-x^2}\,dx\right|$

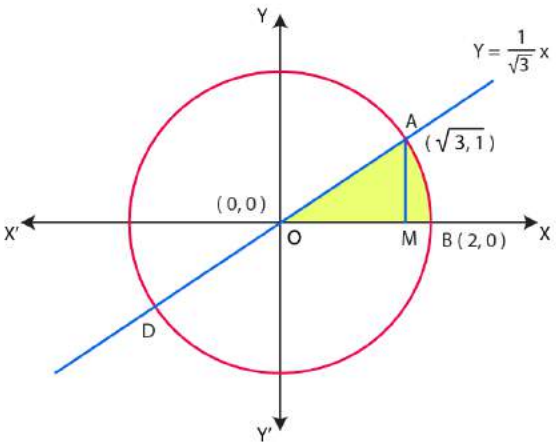
= $6\left[\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^4$ $\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

= $6\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1-\left(0+2\sin^{-1}0\right)\right]$

= $6\left[0+2.\frac{\pi}{2}-0\right]=6\pi$ sq. units

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4.$

Solution:
Step 1: To draw the graphs and shade the region whose area we are to find.



Equation of the circle is $x^2 + y^2 = 2^2$ ……….(1)

We know that equation (1) represents a circle whose centre is (0, 0) and radius is 2.

Equation of the given line is $x = \sqrt{3}y$

$\Rightarrow y = \frac{1}{\sqrt{3}}x$ ……….(2)

We know that equation (2) being of the form $y = mx$ where $m = \frac{1}{\sqrt{3}} = \tan 30^\circ = \tan\theta$

$\Rightarrow \theta = 30^\circ$ represents a straight line passing through the origin and making angle of 30° with x-axis.

Step 2: To find values of x and y.

Put $y = \frac{x}{\sqrt{3}}$ from equation (2) in equation (1),

$x^2 + \frac{x^2}{3} = 4 \Rightarrow 3x^2 + x^2 = 12 \Rightarrow 4x^2 = 12$

$\Rightarrow x^2 = 3 \Rightarrow x = \pm 3$

Putting $x = \pm 3$ in $y = \frac{x}{\sqrt{3}}$, $y = 1$ and $y = -1$

Therefore, the two points of intersections of circle (1) and line (2) are A$\left(\sqrt{3},1\right)$ and D$\left(-\sqrt{3},-1\right)$.

Step 3: Now shaded area OAM between segment OA of line (2) and x-axis

= $\left|\int_0^{\sqrt{3}} y\,dx\right|$ $\left[\because \text{ At O, } x=0 \text{ and at A, } x=\sqrt{3}\right]$

= $\left|\int_0^{\sqrt{3}} \frac{1}{\sqrt{3}} x\,dx\right|$ = $\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}$ = $\frac{1}{\sqrt{3}}\left(\frac{3}{2}-0\right)$ = $\frac{3}{2\sqrt{3}}$ = $\frac{\sqrt{3}}{2}$ sq. units……….(3)

Step IV. Now shaded area AMB between are AB of circle and $x-$ axis.

= = $\left|\int_{\sqrt{3}}^{2} y\,dx\right|$ $\left[\because \text{ At O, } x=\sqrt{3} \text{ and at A, } x=2\right]$

= $\left|\int_{\sqrt{3}}^{2} \sqrt{2^2-x^2}\,dx\right|$ From equation (2),

= $\left(\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_{\sqrt{3}}^{2}$ = $\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1-\left(\frac{\sqrt{3}}{2}\sqrt{4-3}+2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

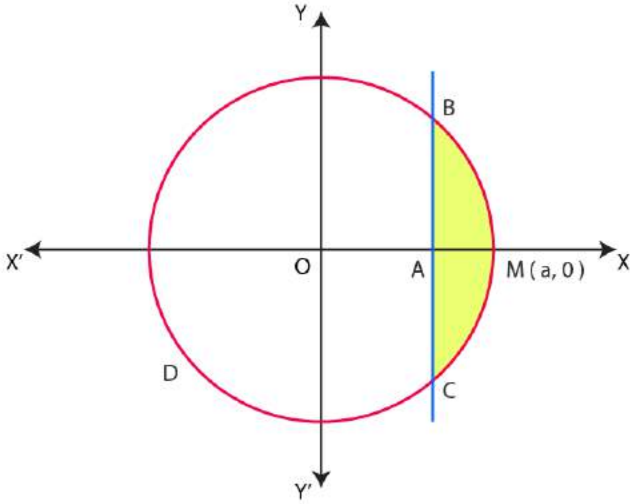
= $0+2.\frac{\pi}{2}-\frac{\sqrt{3}}{2}-2.\frac{\pi}{3}$ = $\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}$ = $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units……….(iv)

Step V. Required shaded area OAB = Area of OAM + Area of AMB

= $\frac{\sqrt{3}}{2}+\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ sq. units

7. Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.

Solution: Equation of the circle is $x^2+y^2=a^2$ ……….(1)



$\therefore\ y^2 = a^2 - x^2$

$\Rightarrow\ y = \sqrt{a^2 - x^2}$ ....(2)

Here,

Area of smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

= Area of ABMC = 2 x Area of ABM

$= 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} y\,dx\right| = 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx\right|$ [From equation (2)]

$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$

$= 2\left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}1 - \left(\frac{\frac{a^2}{\sqrt{2}}}{2}\sqrt{a^2 - \frac{a^2}{2}}\ \sin^{-1}\frac{\frac{a^2}{\sqrt{2}}}{2}\right)\right]$

$= 2\left[0+\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{a}{2\sqrt{2}}\sqrt{\frac{a^2}{2}}-\frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= 2\left[\frac{\pi a^2}{4}-\frac{a}{2\sqrt{2}}\frac{a}{\sqrt{2}}-\frac{a^2}{2}\frac{\pi}{4}\right]$

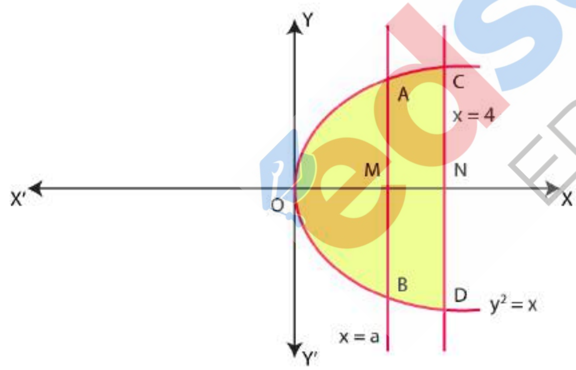
$= 2\left[\frac{\pi a^2}{4}-\frac{\pi a^2}{8}-\frac{a^2}{4}\right]$

$= 2a^2\left[\frac{2\pi-\pi-2}{8}\right]$

$= \frac{a^2}{4}(\pi-2)=\frac{a^2}{4}\left(\frac{\pi}{2}-1\right)$ sq. units

8. The area between $x=y^2$ and x=4 is divided into two equal parts by the line x=a find the value of a.

Solution: Equation of the curve (parabola) is $x=y^2$ ......(1)



$\Rightarrow y=\sqrt{x}$

Now area bounded by parabola (1) and vertical line x=4 is divided into two equal parts by the vertical line x=a.

Area OAMB = Area AMBDNC

$\Rightarrow 2\left|\int_0^a y\,dx\right|=2\left|\int_a^4 y\,dx\right|$

$\Rightarrow 2\left|\int_0^a x^{\frac{1}{2}}\,dx\right| = 2\left|\int_a^4 x^{\frac{1}{2}}\,dx\right|$

$\Rightarrow \dfrac{\left(x^{\frac{3}{2}}\right)_0^a}{\frac{3}{2}} = \dfrac{\left(x^{\frac{3}{2}}\right)_a^4}{\frac{3}{2}}$

$\Rightarrow \frac{2}{3}\left[a^{\frac{3}{2}} - 0\right] = \frac{2}{3}\left[4^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$

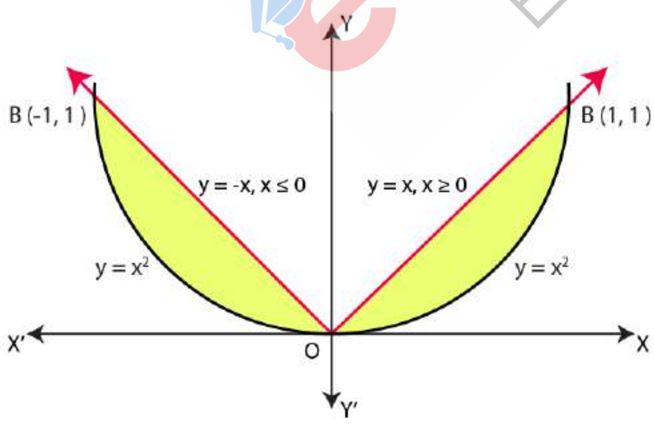
$\Rightarrow a^{\frac{3}{2}} = 8 - a^{\frac{3}{2}}$

$\Rightarrow 2a^{\frac{3}{2}} = 8 \Rightarrow a^{\frac{3}{2}} = 4$

$\Rightarrow a = 4^{\frac{2}{3}}$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solution: The required area is the area included between the parabola $y = x^2$ and the modulus function $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$



To find: Area between the parabola $y = x^2$ and the ray $y = x$ for $x \geq 0$

Here, Limits of integration $\Rightarrow$ $y=x$

$\Rightarrow$ $x^2=x$ $\Rightarrow x^2-x=0$

$\Rightarrow$ $x(x-1)=0$ $\Rightarrow x=0, x=1$

Now, for $y=|x|$, table of values,

$y=x$ if $x\geq 0$

x	0	1	2
y	0	1	2

$y=-x$ if $x\leq 0$

x	0	-1	-2
y	0	1	2

Now, Area between parabola $y=x^2$ and x-axis between limits x=0 and x=1

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$ ………..(1)

And Area of ray y=x and x-axis,

$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$ ………..(2)

So, Required shaded area in first quadrant

= Area between ray y=x for $x\geq 0$ and x-axis – Area between parabola $y=x^2$ and x-axis in first quadrant

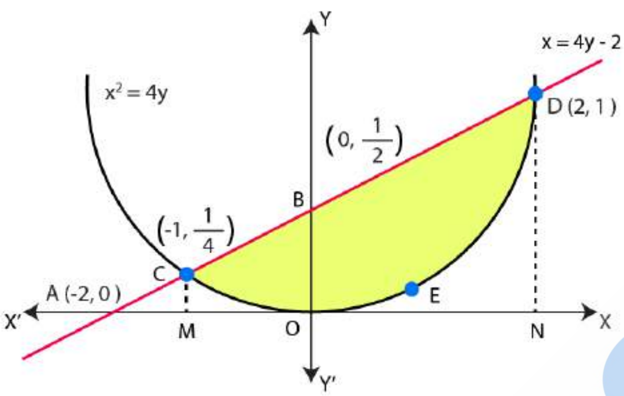
= Area given by equation (2) – Area given by equation (1)

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

10. Find the area bounded by the curve $x = 4y$ and the line $x = 4y - 2$.

Solution:

Step I. Graphs and region of Integration



Equation of the given curve is

$x^2 = 4y$ ……..(1)

Equation of the given line is

$x = 4y - 2$ ……….(2)

$\Rightarrow y = \frac{x+2}{4}$

x	0	1	-2
y	0	1/2	0

Step 2: Putting $y = \frac{x^2}{4}$ from equation (1) in equation (2),

$x = 4.\frac{x^2}{4} - 2 \Rightarrow x = x^2 - 2 \Rightarrow -x^2 + x + 2 = 0$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow$

$x^2 - 2x + x - 2 = 0 \Rightarrow x(x-2) + (x-2) = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

For x=2,from equation (1), $y = \frac{x^2}{4} = \frac{4}{4} = 1$

So point is (2, 1)

For x=-1 from equation (1), $y = \frac{x^2}{4} = \frac{1}{4}$

So point is $\left(-1, \frac{1}{4}\right)$

Therefore, the two points of intersection of parabola (1) and line (2) are C $\left(-1, \frac{1}{4}\right)$ and D (2, 1).

Step 3. Area CMOEDN between parabola (1) and x-axis

$= \left|\int_{-1}^{2} y\, dx\right| = \left|\int_{-1}^{2} \frac{x^2}{4}\, dx\right|$

$= \left|\frac{\left(x^3\right)_{-1}^{2}}{12}\right| = \left|\frac{1}{12}\left\{2^3 - (-1)^3\right\}\right|$

$= \frac{1}{12}(8+1) = \frac{9}{12} = \frac{3}{4}$ sq. units……..(3)

Step 4. Area of trapezium CMND between line (2) and x-axis

$= \left|\int_{-1}^{2} y\, dx\right| = \left|\int_{-1}^{2} \frac{x+2}{4}\, dx\right|$

$= \left|\frac{1}{4}\int_{-1}^{2} (x+2)\, dx\right| = \frac{1}{4}\left|\left(\frac{x^2}{2} + 2x\right)_{-1}^{2}\right|$

$= \frac{1}{4}\left|\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right| = \frac{1}{4}\left|\left(2 + 4 - \frac{1}{2} + 2\right)\right|$

$= \frac{1}{4}\left|8-\frac{1}{2}\right| = \frac{1}{4}\times\frac{15}{2}=\frac{15}{8}$ sq. units……….(4)

Therefore,

Required shaded area = Area given by equation (4) – Area given by equation (3)

$= \frac{15}{8}-\frac{3}{4}=\frac{15-6}{8}=\frac{9}{8}$ sq. units

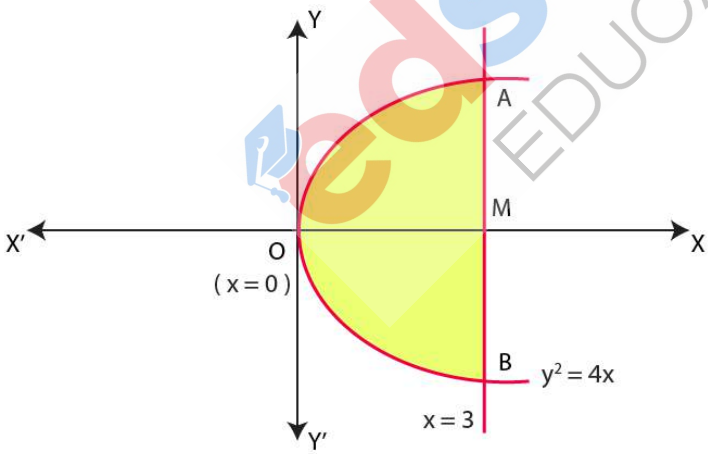
11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line x=3.

Solution: Equation of the (parabola) curve is

$y^2 = 4x$ ………(1)

$\Rightarrow y = 4x = 2x^{\frac{1}{2}}$ ………(2)

Here required shaded area OAMB = 2 x Area OAM



$= 2\left|\int_0^3 y\,dx\right| = 2\left|\int_0^3 2x^{\frac{1}{2}}\,dx\right| = 4\left|\frac{\left(x^{\frac{3}{2}}\right)_0^3}{\frac{3}{2}}\right|$

$= 4.\frac{2}{3}\left[3^{\frac{3}{2}} - 0\right] = \frac{8}{3}.3\sqrt{3} = 8\sqrt{3}$ sq. units

12. Choose the correct answer:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines x = 0 and x = 2 is

(A) π **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$

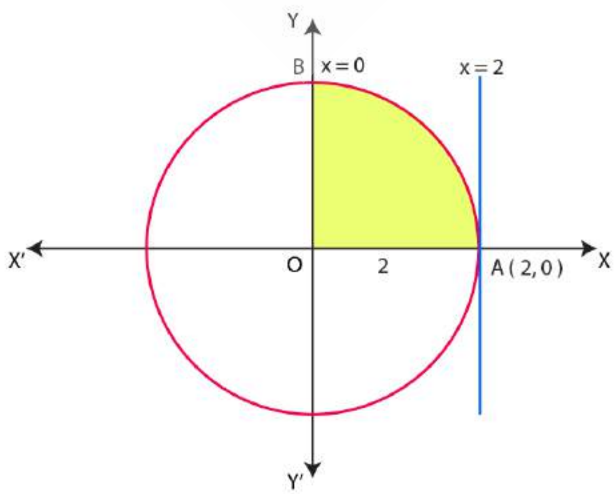
Solution:

Option (A) is correct.

Explanation:

Equation of the circle is $x^2 + y^2 = 2^2$ ……….(1)

$\Rightarrow y = \sqrt{2^2 - x^2}$ ……….(2)



Required area = $\left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2 - x^2}\,dx\right|$

= $\left|\left(\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2\right|$

= $\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1-\left(0+2\sin^{-1}0\right)$

= $0+2.\frac{\pi}{2}-0-0$ = π sq. units

13. Choose the correct answer:

Area of the region bounded by the curve $y^2 = 4x$, y- axis and the line $y = 3$ is:

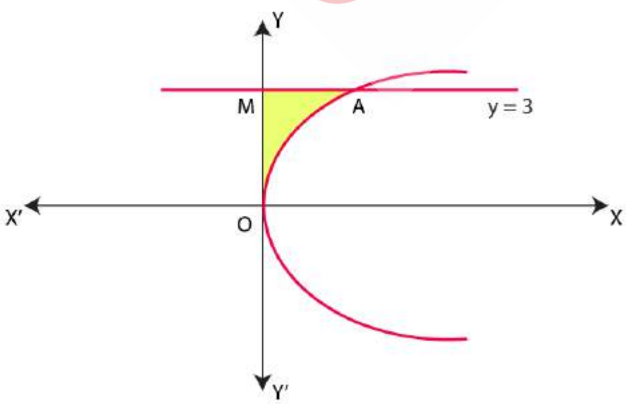
(A) 2 (B) 9/4 (C) 9/3 (D) 9/2

Solution:

Option (B) is correct.

Explanation:

Equation of the curve (parabola) is $y^2 = 4x$



Required area = Area OAM = $\left|\int_0^3 x\,dy\right|$ = $\left|\int_0^2 \frac{y^2}{4}\,dy\right|$

= $\frac{1}{4}\left|\left(\frac{y^3}{3}\right)_0^2\right|$ = $\frac{1}{4}\left|\frac{27}{3}-0\right|=\frac{9}{4}$ sq. units

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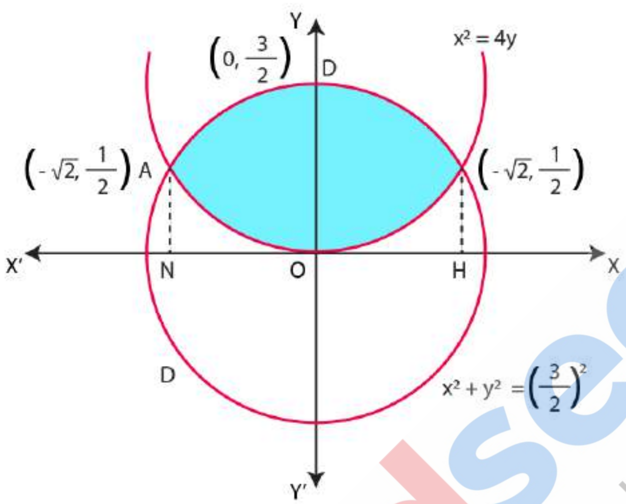
1. Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$.

Solution:

Step 1: Equation of the circle is $4x^2+4y^2=9$

$x^2+y^2=\frac{9}{4}$ ....(1)



Here, centre of circle is (0, 0) and radius is 3/2

Equation of parabola is $x^2=4y$ .....(2)

Step 2: To find values of x and y

Put $x^2=4y$ in equation (1), $4y+y^2=\frac{9}{4}$

$16y+4y^2=9$

$4y^2+16y-9=0$

$4y^2+18y-2y-9=0$

$2y(2y+9)-1(2y+9)=0$

$(2y+9)(2y-1)=0$

$2y+9=0$ or $2y-1=0$

$\Rightarrow y=\frac{-9}{2}$ or $y=\frac{1}{2}$

Find the value of x:

Put $y=\frac{-9}{2}$ in $x^2=4y$,

$\Rightarrow x^2=4\left(\frac{-9}{2}\right)=-18$

Put $y=\frac{1}{2}$ in $x^2=4y$,

$\Rightarrow x^2=4\left(\frac{1}{2}\right)=2$

=> $x=\pm 2$

Therefore, Points of intersections of circle (1) and parabola (2) are

A $\left(-\sqrt{2},\frac{1}{2}\right)$ and B $\left(\sqrt{2},\frac{1}{2}\right)$.

Step 3: Area OBM = Area between parabola (2) and y-axis

$= \int_0^{\frac{1}{2}} x\,dy$

$\left[\because \text{At O}, y=0 \text{ and at B}, y=\frac{1}{2}\right]$

$= \int_0^{\frac{1}{2}} 2y^{\frac{1}{2}}\,dy$

$$\left[\because x^2 = 4y \Rightarrow x = 2\sqrt{y} = 2y^{\frac{1}{2}}\right]$$

$$= \frac{2.\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}} = 2.\frac{2}{3}\left[\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right]$$

$$= \frac{4}{3}.\frac{1}{2}\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{3}$$ ....(3)

Step 4: Now area BDM = Area between circle (1) and y-axis

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} x\, dy$$

$$\left[\because \text{At B}, y = \frac{1}{2} \text{ and at D}, y = \frac{3}{2}\right]$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} \left(\frac{3}{2}\right)^2 - y^2\, dy$$

$$\left[\because x^2 = \left(\frac{3}{2}\right)^2 - y^2 \Rightarrow x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\right]$$

$$= \left[\frac{y}{2}\sqrt{\left(\frac{3}{2}\right)^2 - y^2} + \frac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\frac{y}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{3}{4}\sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} + \frac{9}{8}\sin^{-1}\frac{\frac{3}{2}}{\frac{3}{2}} - \left[\frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8}\sin^{-1}\frac{\frac{1}{2}}{\frac{3}{2}}\right]$$

$= \left(\frac{3}{4}\times 0\right)+\frac{9}{8}\sin^{-1}1-\left[\frac{1}{4}\sqrt{\frac{8}{4}}+\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$

$= \frac{9}{8}\times\frac{\pi}{2}-\frac{1}{4}\sqrt{2}-\frac{9}{8}\sin^{-1}\frac{1}{3}$

$= \frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}$ …….(4)

Step 5.
Required shaded area = Area AOBDA = 2 (Area OBD) = 2 (Area OBM + Area MBD)

$= 2\left[\frac{\sqrt{2}}{3}+\left(\frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right] = 2\left[\sqrt{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$

$= 2\sqrt{2}\left(\frac{4-1}{12}\right)+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

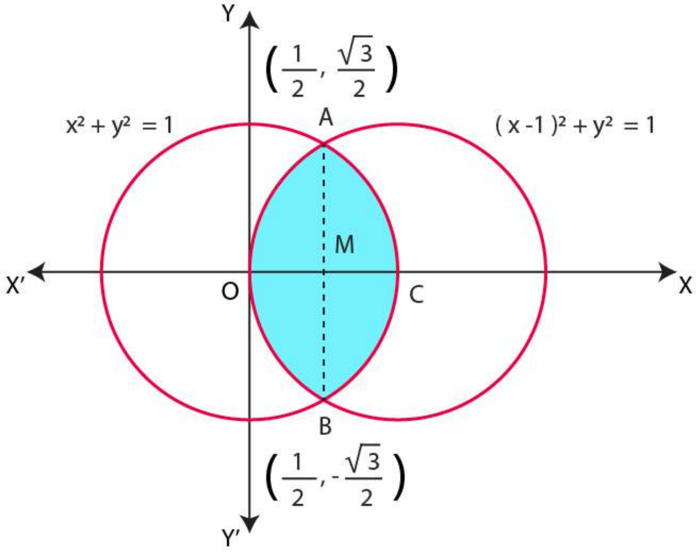
$= \frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{\sqrt{2}}{6}+\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$

$= \frac{\sqrt{2}}{6}+\frac{9}{4}\cos^{-1}\frac{1}{3}\quad\left[\because \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\right]$

2. Find the area bounded by the curves $(x-1)^2+y^2=1$ **and** $x^2+y^2=1$.

Solution:
Equations of two circles are

$x^2+y^2=1$ ……….(1)



And $(x-1)^2 + y^2 = 1$ ....(2)

From equation (1), $y^2 = 1 - x^2$

Put this value in equation (2),

$(x-1)^2 + 1 - x^2 = 1$

$\Rightarrow x^2 + 1 - 2x + 1 - x^2 = 1$

$\Rightarrow -2x + 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Put $x = \frac{1}{2}$ in $y^2 = 1 - x^2$,

$y^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{4}$

The two points of intersections of circles (1) and (2) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

Now, from equation (1), $y=\sqrt{1-x^2}$ in first quadrant and from equation (2), $y=\sqrt{1-(x-1)^2}$ in first quadrant.

Required area OACBO = 2 x Area OAC = 2 (Area OAD + Area DAC)

$$= 2\left[\int_0^{\frac{1}{2}} y \text{ of circle (ii) } dx + \int_{\frac{1}{2}}^{1} y \text{ of circle (i) } dx\right]$$

$$= 2\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\, dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\, dx\right]$$

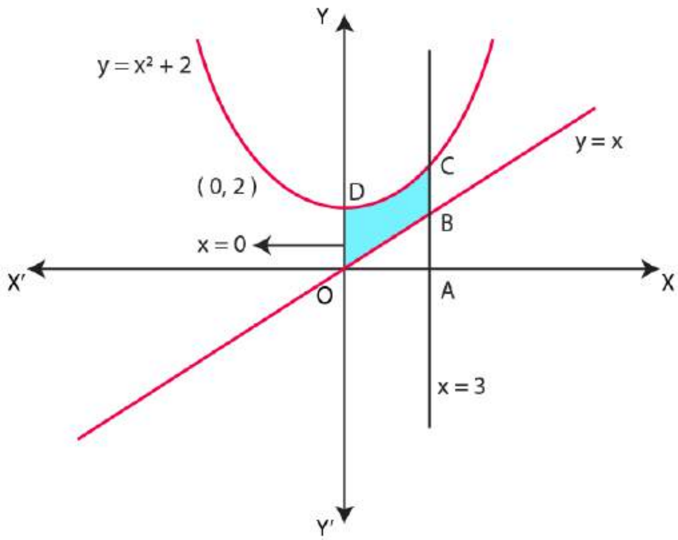
$$= 2\left[\left\{\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}(x-1)\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right\}_{\frac{1}{2}}^{1}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \sin^{-1}(-1) - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

3. Find the area of the region bounded by the curves $y=x^2+2, y=x, x=0$ and x=3.

Solution: Equation of the given curve is



(Point D is (0,2))

$y = x^2 + 2$ ....(1)

$x^2 = y - 2$

Here Vertex of the parabola is (0, 2).

Equation of the given line is y=x ....(2)

x	0	1	2
y	0	1	2

We know that, slope of straight line passing through the origin is always 1, that means, making an angle of 45 degrees with x- axis.

Here also, Limits of integration area given to be x=0 to x=3.

Area bounded by parabola (1) namely $y = x^2 + 2$, the x-axis and the ordinates x=0 to x=3 is the area OACD and $\int_0^3 y\, dx = \int_0^3 (x^2 + 2)\, dx$

$= \left(\frac{x^3}{3} + 2x\right)_0^3$

= (9 + 6) – 0 = 15 ……….(3)

Again Area bounded by parabola (2) namely y=x the x-axis and the ordinates x=0 to x=3 is the area OAB and

$\int_0^3 y\,dx = \int_0^3 x\,dx$

$= \left(\frac{x^2}{2}\right)_0^3 = \frac{9}{2} - 0 = \frac{9}{2}$ ……….(3)

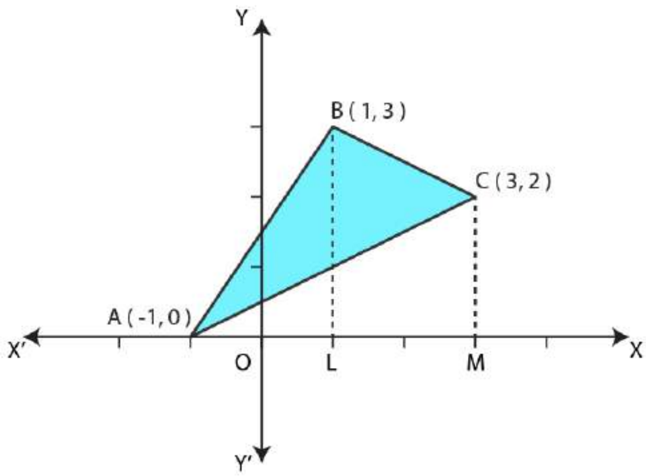
Required area = Area OBCD = Area OACD – Area OAB

= Area given by equation (3) – Area given by equation (4)

$= 15 - \frac{9}{2} = \frac{21}{2}$ sq. units

4. Using integration, find the area of the region bounded by the triangle whose vertices are (-1, 0), (1, 3) and (3, 2).

Solution:
Vertices of triangle are A(-1, 0), B (1, 3) and C (3, 2).



Therefore, eequation of the line is

$$y-0=\frac{3-0}{1-(-1)}(x-(-1))$$

$$\left[\because y-y_1=\frac{y_2-y_1}{x_2-x_1}(x_2-x_1)\right]$$

$$y=\frac{3}{2}(x+1)$$

Area of ΔABL = Area bounded by line AB and x-axis

$$=\int_{-1}^{1} y\,dx$$

$$[\because \text{At A}, x=-1 \text{ and at B}, x=1]$$

$$=\int_{-1}^{1}\frac{3}{2}(x+1)\,dx$$

$$=\frac{3}{2}\left(\frac{x^2}{2}+x\right)_{-1}^{1}$$

$= \frac{3}{2}\left[\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right]$

$= \frac{3}{2}\left(\frac{3}{2}+\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{4}{2}=3$ ……….(1)

Again equation of line BC is $y-3=\frac{2-3}{3-1}(x-1) \Rightarrow y=\frac{1}{2}(7-x)$

Area of trapezium BLMC = Area bounded by line BC and x-axis

⇨ $\int_1^3 y\,dx = \int_1^3 \frac{1}{2}(7-x)\,dx$

$= \frac{1}{2}\left(7x-\frac{x^2}{2}\right)_1^3$

$= \frac{1}{2}\left[\left(21-\frac{9}{2}\right)-\left(7-\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(21-\frac{9}{2}-7+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{42-9-14+1}{2}\right)$

$= \frac{1}{4}\times 20$ = 5 ……….(2)

Again equation of line AC is $y-0=\frac{2-0}{3-(-1)}(x-(-1)) \Rightarrow y=\frac{1}{2}(x+1)$

Area of triangle ACM = Area bounded by line AC and x-axis

=> $\int_{-1}^3 y\,dx = \int_{-1}^3 \frac{1}{2}(x+1)\,dx$

$= \frac{1}{2}\left[\left(\frac{x^2}{2}+x\right)_{-1}^{3}\right]$

$= \frac{1}{2}\left(\frac{9}{2}+3-\frac{1}{2}+1\right)$

$= \frac{1}{2}\left(\frac{9+6-1+2}{2}\right)$

$= \frac{1}{4}\times 16$ = 4 ……….(3)

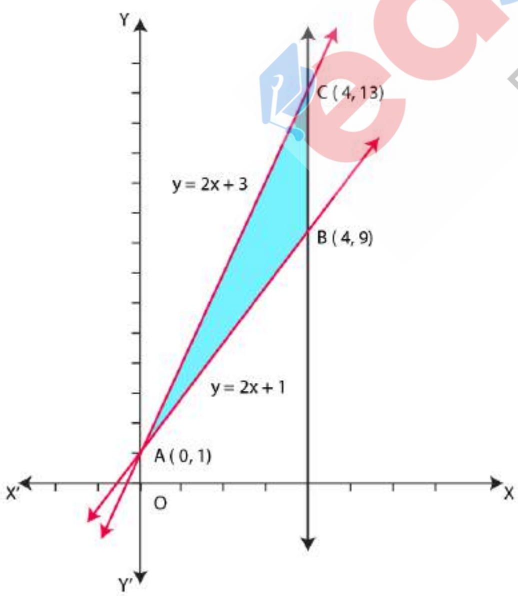
Therefore,

Required area = Area of ΔABL + Area of Trapezium BLMC – Area of ΔACM

= 3 + 5 – 4 = 4 sq. units

5. Using integration, find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x=4.

Solution: Equations of one side of triangle is



$y = 2x + 1$ ....(1)

$y = 3x + 1$ ....(2) and

x = 4 ....(3)

Solving equation (1) and (2), we get x=0 and y=1

So, Point of intersection of lines (1) and (2) is A (0, 1)

Put x=4 in equation (1), we get y=9

So, Point of intersection of lines (1) and (3) is B (4, 9)

Put x=4 in equation (1), we get y=13

Point of intersection of lines (2) and (3) is C (4, 13)

Area between line (2), that is AC and x-axis

= $\int_0^4 y\,dx$ = $\int_0^4 (3x+1)\,dx$ = $\left(\frac{3x^2}{2} + x\right)_0^4$

= 24 + 4 = 28 sq. units ....(iv)

Again Area between line (1) , that is AB and x-axis

= $\int_0^4 y\,dx$ = $\int_0^4 (2x+1)\,dx$

= $\left(x^2 + x\right)_0^4$

= 16 + 4 = 20 sq. units ....(v)

Therefore, Required area of ΔABC

= Area given by (4) – Area given by (5)

= 28 – 20 = 8 sq. units

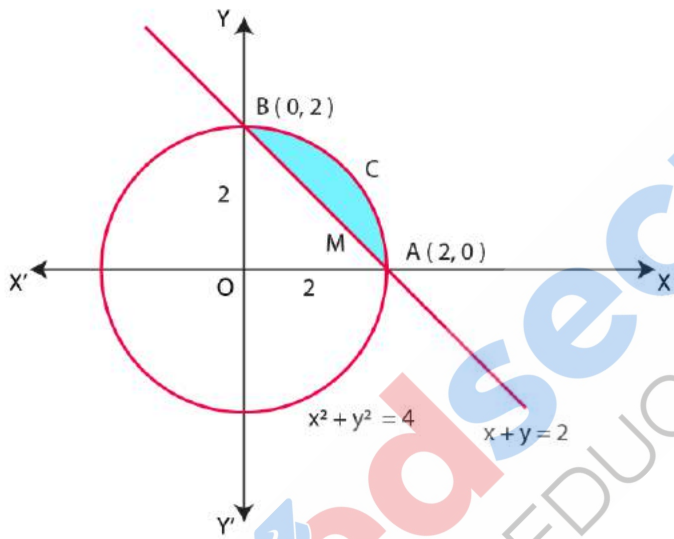
6. Choose the correct answer:

Smaller area enclosed by the circle $x^2+y^2=4$ **and the line** $x+y=2$ **is:**

(A) $2(\pi-2)$ **(B)** $\pi-2$ **(C)** $2\pi-1$ **(D)** $2(\pi+2)$

Solution:
Option (B) is correct.

Explanation:
Equation of circle is $x^2+y^2=2^2$ ……….(1)



$\Rightarrow y=\sqrt{2^2-x^2}$ ………(2)

Also, equation of the line is $x+y=2$ ……….(3)

or y = 2-x

x	0	2
y	2	0

Therefore graph of equation (3) is the straight line joining the points (0, 2) and (2, 0).

From the graph of circle (1) and straight line (3), it is clear that points of intersections of circle (1) and straight line (3) are A (2, 0) and B (0, 2).

Area OACB, bounded by circle (1) and coordinate axes in first quadrant

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2 - x^2}\,dx\right|$

$= \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2$

$= \left(\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}1\right) - \left(0 + 2\sin^{-1}0\right)$

$= 0 + 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$ sq. units ……….(iv)

Area of triangle OAB, bounded by straight line (3) and coordinate axes

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 (2-x)\,dx\right|$

$= \left(2x - \frac{x^2}{2}\right)_0^2$

$= (4-2) - (0-0)$ = 2 sq. units ……….(v)

Required shaded area = Area OACB given by (iv) – Area of triangle OAB by (v)

$= (\pi - 2)$ sq. units

7. Choose the correct answer:

Area lying between the curves $y^2 = 4x$ and $y = 2x$ is:

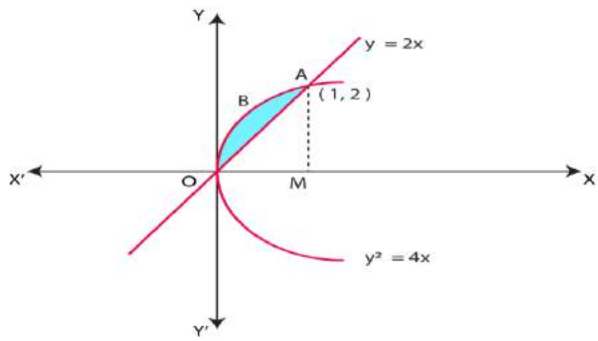
(A) $\frac{2}{3}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$

Solution:

Option (B) is correct.

Explanation:

Equation of curve (parabola) is $y^2 = 4x$ …..(1)



$\Rightarrow y = 2\sqrt{x} = 2x^{\frac{1}{2}}$ .....(2)

Equation of another curve (line) is y=2x .....(3)

Solving equation (1) and (3), we get x=0 or x=1 and y=0 or y=2

Therefore, Points of intersections of circle (1) and line (2) are O (0, 0) and A (1, 2).

Now Area OBAM = Area bounded by parabola (1) and x-axis = $\left|\int_0^1 y\, dx\right|$

$= \left|\int_0^1 2x^{\frac{1}{2}}\, dx\right| = 2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}}$

$= \frac{4}{3}(1-0) = \frac{4}{3}$ ....(4)

Also, Area ΔOAM = Area bounded by parabola (3) and x-axis

$= \left|\int_0^1 y\, dx\right| = \left|\int_0^1 2x\, dx\right| = 2\left(\frac{x^2}{2}\right)_0^1$

$= (1-0) = 1$ ....(5)

Now required shaded area OBA = Area OBAM – Area of ΔOAM

$= \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$ sq. units

Miscellaneous Examples

1. Find the area under the given curves and given lines:

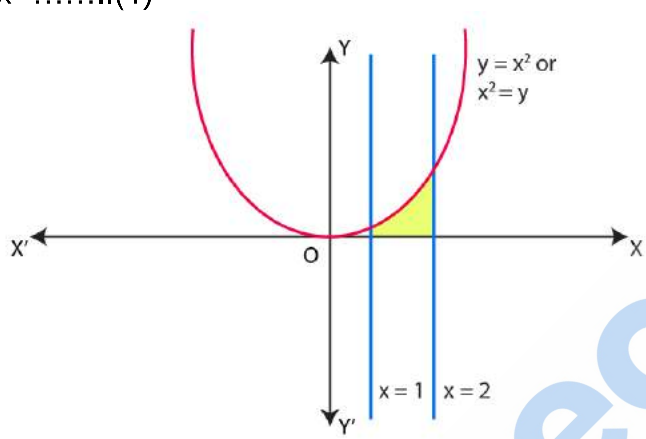
(i) $y = x^2, x = 1, x = 2$ **and x-axis.**

(ii) $y = x^4, x = 1, x = 5$ **and x-axis.**

Solution:

(i) Equation of the curve is

y= x^2 ……..(1)



Required area bounded by curve (1), vertical line x=1, x=2 and x-axis

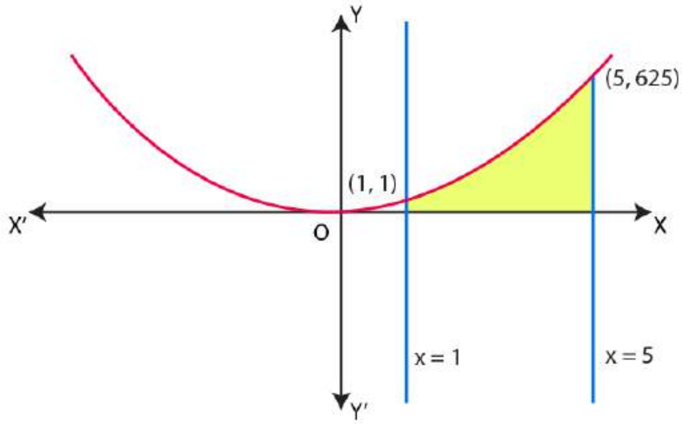
$= \int_1^2 y\, dx$

$= \left(\frac{x^3}{3}\right)_1^2$

$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ sq. units

(ii) Equation of the curve

y= x^4 …..(1)



It is clear that curve (1) passes through the origin because x=0 from (1) y=0.

Table of values for curve $y = x^4$

x	1	2	3	4	5
y	1	16	81	256	625

Required shaded area between the curve $y = x^4$, vertical lines $x = 1, x = 5$ and $x-$ axis

$= \int_1^5 y\,dx = \int_1^5 x^4\,dx$

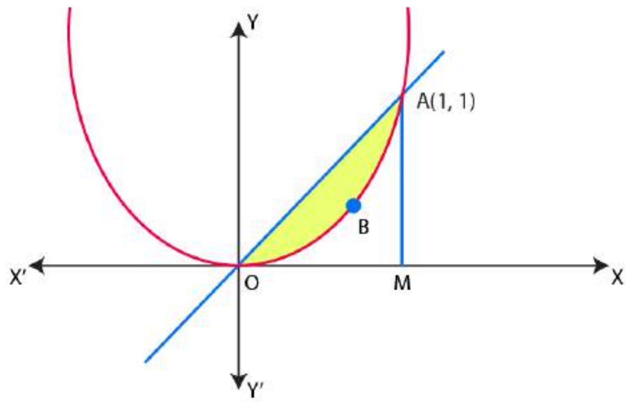
$= \left(\frac{x^5}{5}\right)_1^5 = \frac{5^5}{5} - \frac{1^5}{5}$

$= \frac{3125-1}{5} = \frac{3124}{5}$

= 624.8 sq. units

2. Find the area between the curves y=x and y=x^2

Solution: Equation of one curve (straight line) is y=x …..(i)



Equation of second curve (parabola) is $y = x^2$ .....(ii)

Solving equation (i) and (ii), we get $x = 0$ or $x = 1$ and $y = 0$ or $y = 1$

Points of intersection of line (i) and parabola (ii) are O (0, 0) and A (1, 1).

Now Area of triangle OAM

= Area bounded by line (i) and x-axis

$= \int_0^1 y\,dx = \int_0^1 x\,dx$

$= \left(\frac{x^2}{2}\right)_0^1$

$= \frac{1}{2} - 0 = \frac{1}{2}$ sq. units

Also Area OBAM = Area bounded by parabola (ii) and x-axis

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx$

$= \left(\frac{x^3}{3}\right)_0^1$

= $\frac{1}{3} - 0 = \frac{1}{3}$ sq. units

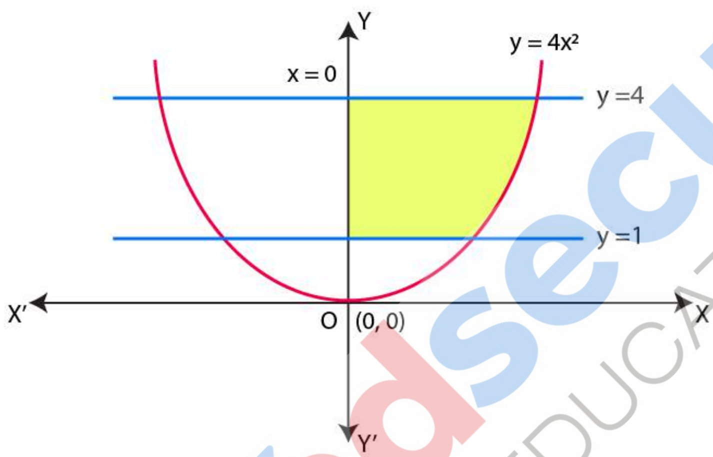
Required area OBA between line (i) and parabola (ii)

= Area of triangle OAM – Area of OBAM

= $\frac{1}{2} - \frac{1}{3}$ = $\frac{3-2}{6} = \frac{1}{6}$ sq. units

3. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and y=4.

Solution: Equation of the curve is $y = 4x^2$



$x^2 = \frac{y}{4}$ ……….(i)

or $x = \frac{\sqrt{y}}{2}$ ……….(ii)

Here required shaded area of the region lying in first quadrant bounded by parabola (i), $x = 0$ and the horizontal lines $y = 1$ and $y = 4$ is

$\int_1^4 x\,dy$ = $\int_1^4 \frac{\sqrt{y}}{2}\,dy$ $= \frac{1}{2}\int_1^4 y^{\frac{1}{2}}$

$= \frac{1}{2}\left|\frac{\left(y^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$

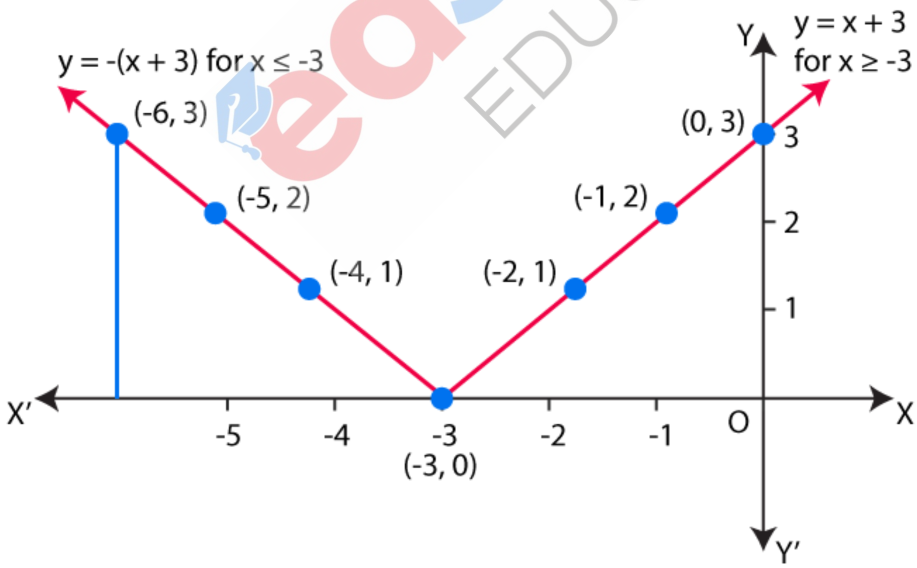
$= \frac{1}{2}\cdot\frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$

$= \frac{1}{3}\left(4\sqrt{4} - 1\right)$

$= \frac{1}{3}(8-1)$ $= \frac{7}{3}$ sq. units

4. Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3|\, dx$.

Solution: Equation of the given curve is $y = |x+3|$ ....(i)



$y = |x+3| \geq 0$ for all real x.

Graph of curve is only above the x-axis i.e., in first and second quadrant only.

$y = |x+3|$

$= x+3$

If $x+3 \geq 0$

$x \geq -3$ ….(ii)

And $y = |x+3|$

$= -(x+3)$

If $x+3 \leq 0$

$x \leq -3$ …….(iii)

Table of values for $y = x+3$ for $x \geq -3$

x	y
-3	0
-2	1
-1	2
0	3

Table of values for $y = x+3$ for $x \leq -3$

x	y
-3	0
-4	1
-5	2
-6	3

Now, $\int_{-6}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} |x+3|\, dx + \int_{-3}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} -(x+3)\, dx + \int_{-3}^{0} (x+3)\, dx$

$= \left(\frac{x^2}{2}+3x\right)_{-6}^{-3} + \left(\frac{x^2}{2}+3x\right)_{-3}^{0}$

$= \left[\frac{9}{2}-9-(18-18)\right] + \left[0-\left(\frac{9}{2}-9\right)\right]$

$= \frac{9}{2}+9+0+0-\frac{9}{2}+9$

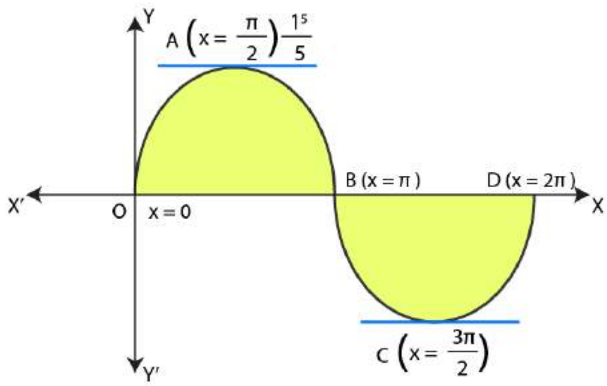
$= 18-\frac{18}{2} = 18-9 = 9$ sq. units

5. Find the area bounded by the curve y = sin x between x=0 and x=2π.

Solution: Equation of the curve is y = sin x ……….(i)

$y = \sin x \geq 0$ for $0 \leq x \leq \pi$: as graph is in I and II quadrant.

And $y = \sin x \leq 0$ for $\pi \leq x \leq 2\pi$: as graph is in III and IV quadrant.



If tangent is parallel to x-axis, then

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Table of values for curve y= sin x between x =0 and $x = 2\pi$

x	y
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Now Required shaded area = Area OAB + Area BCD

$= \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} y\,dx$

$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} \sin x\,dx$

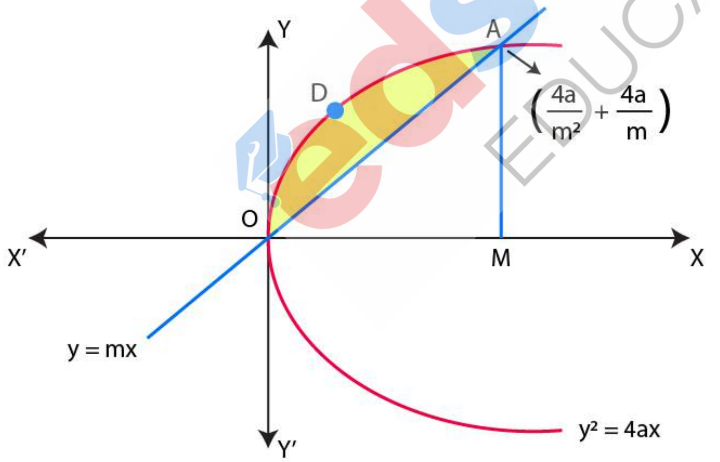
$= -(\cos x)_0^{\pi} + (\cos x)_{\pi}^{2\pi}$

$= -1(-1-1) + -(1+1)$

= 2 + 2 = 4 sq. units

6. Find the area enclosed by the parabola $y^2 = 4ax$ and the line y=mx.

Solution: Equation of parabola is $y^2 = 4ax$ ……..(i)



The area enclosed between the parabola and line is the shaded area OADO.

Form figure: And the points of intersection of curve and line are

O (0, 0) and A $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

Now Area ODAM = Area of parabola and x-axis

$= \int_0^{\frac{4a}{m^2}} 2\sqrt{a}\, x^{\frac{1}{2}}\, dx$

$= 2\sqrt{a}\, \frac{\left(x^{\frac{3}{2}}\right)_0^{\frac{4a}{m^2}}}{\frac{3}{2}}$

$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}}$

$= \frac{32a^2}{3m^3}$ ……….(ii)

Again Area of ΔOAM = Area between line and x-axis

$= \left| \int_0^{\frac{4a}{m^2}} mx\, dx \right| = m\left(\frac{x^2}{2}\right)_0^{\frac{4a}{m^2}}$

$= \frac{m}{2}\left(\left(\frac{4a}{m^2}\right)^2 - 0\right)$

$= \frac{m}{2} \cdot \frac{16a^2}{m^4} = \frac{8a^2}{m^3}$ ……….(ii)

Requires shaded area = Area ODAM – Area of ΔOAM

$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$

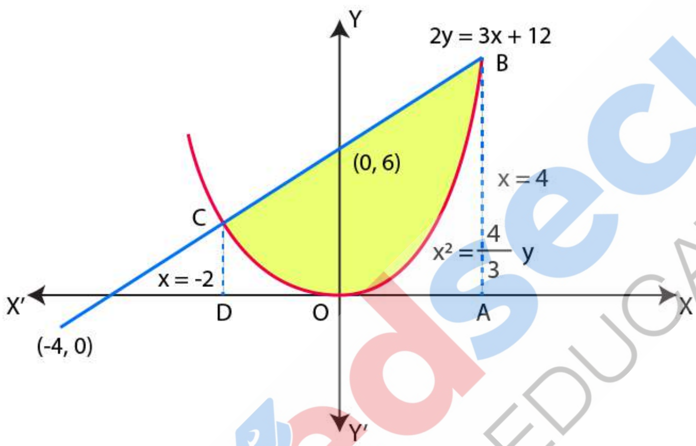
$= \frac{a^2}{m^3}\left(\frac{32}{3}-8\right)$

$= \frac{8a^2}{3m^3}$

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

Solution:
Equation of the parabola is
$4y = 3x^2$ ……….(i)

or $x^2 = \frac{4}{3}y$



Equation of the line is $2y = 3x + 12$ ……(ii)

From graph, points of intersection are B (4, 12) and C(-2, 3).

Now, Area ABCD = $\left|\int_{-2}^{4}\left(\frac{3}{2}x+6\right)dx\right|$

$= \left[\frac{3}{4}x^2+6x\right]_{-2}^{4}$

$= (12+24)-(3-12)$

= 45 sq. units

Again, Area CDO + Area OAB = $\int_{-2}^{4}\left(\frac{3}{4}x^2\right)dx$

= $\frac{1}{4}\left[64-(-8)\right]$ = 18 sq. units

Therefore,

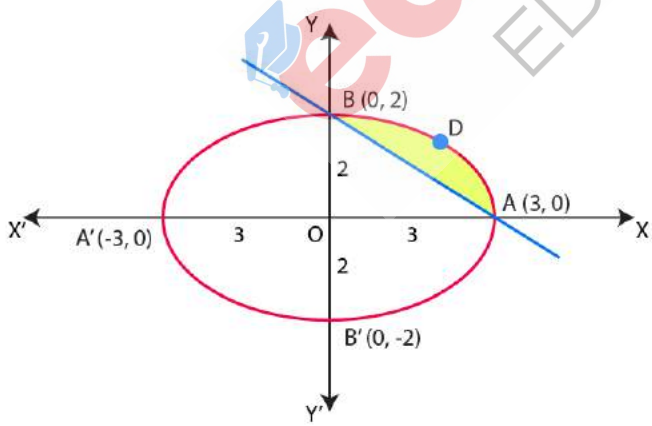
Required area = Area ABCD – (Area CDO + Area OAB)

= 45 – 18 = 27 sq. units

8. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

Solution: Equation of the ellipse is

$\frac{x^2}{9}+\frac{y^2}{4}=1$ ....(i)



Here points of intersection of ellipse (i) with x-axis are

A (3, 0) and A'(-3, 0) and intersection of ellipse (i) with y- axis are B (0, 2) and B'(0, -2).

Also, the points of intersections of ellipse (i) and line $\frac{x}{3}+\frac{y}{2}=1$ are A (3, 0) and B (0, 2).

Therefore,

Area OADB = Area between ellipse (i) (arc AB of it) and x-axis

$= \int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$

$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2-x^2}+\frac{3^2}{2}\sin^{-1}\frac{x}{3}\right]$

$= \frac{2}{3}\left[\frac{3}{2}\sqrt{9-9}+\frac{9}{2}\sin^{-1}1-\left(0+\frac{9}{2}\sin^{-1}0\right)\right]$

$= \frac{2}{3}.\frac{9\pi}{4} = \frac{3\pi}{2}$ sq. units……….(ii)

Again Area of triangle OAB = Area bounded by line AB and x-axis

$= \int_0^3 \frac{2}{3}\sqrt{3-x}\,dx$

$= \frac{2}{3}\left\{\left(9-\frac{9}{2}\right)-0\right\}$

$= \frac{2}{3}.\frac{9}{2}=3$ sq. units ….(iii)

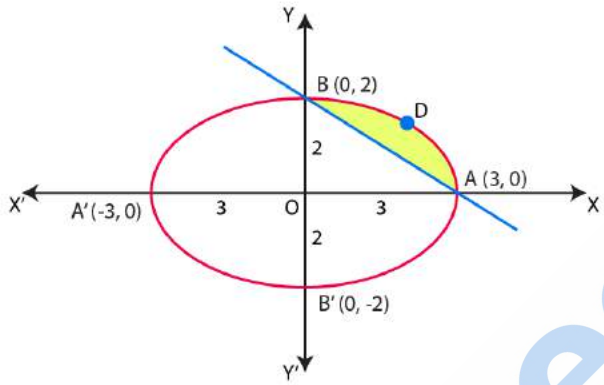
Now Required shaded area = Area OADB – Area OAB

$= \frac{3\pi}{2}-3$

$= 3\left(\frac{\pi}{2}-1\right)=\frac{3}{2}(\pi-2)$ sq. units

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

Solution: Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ……..(i)



Area between arc AB of the ellipse and x-axis

$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{b}{a}\left[0+\frac{a^2}{2}\sin^{-1}1-(0+0)\right]$

$= \frac{b}{a}.\frac{a^2}{2}.\frac{\pi}{2}=\frac{\pi ab}{4}$ ………..(ii)

Also Area between chord AB and x-axis

$= \int_0^a \frac{b}{a}(a-x)\ dx$

$= \frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a$

$= \frac{b}{a}\left(a^2 - \frac{a^2}{2}\right)$

$= \frac{b}{a}.\frac{a^2}{2} = \frac{1}{2}ab$

Now, Required area = (Area between arc AB of the ellipse and x-axis) – (Area between chord AB and x-axis)

$= \frac{\pi ab}{4} - \frac{ab}{2} = \frac{ab}{4}(\pi - 2)$ sq. units

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x+2$ and x- axis.

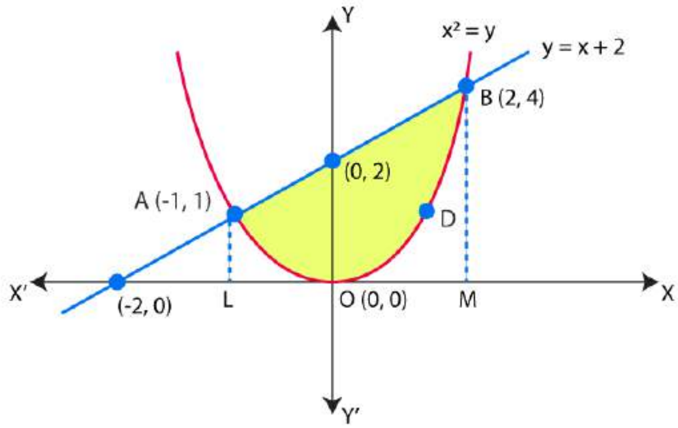
Solution: Equation of parabola is $x^2 = y$ …………(i)
Equation of line is $y = x+2$ …………(ii)

Here the two points of intersections of parabola (i) and line (ii) are A(-1, 1) and B (2, 4).

Area ALODBM = Area bounded by parabola (i) and x-axis

$= \int_{-1}^{2} x^2\ dx = \left(\frac{x^3}{3}\right)_{-1}^{2}$

$= \frac{8}{3} + \frac{1}{3} = \frac{9}{3}$ = 3 sq. units



Also, Area of trapezium ALMB = Area bounded by line (ii) and x-axis

$= \int_{-1}^{2} (x+2)\, dx = \left(\frac{x^2}{2} + 2x\right)_{-1}^{2}$

$= 2 + 4 - \left(\frac{1}{2} - 2\right)$

$= \frac{15}{2}$ sq. units

Now, required area = Area of trapezium ALMB – Area ALODBM

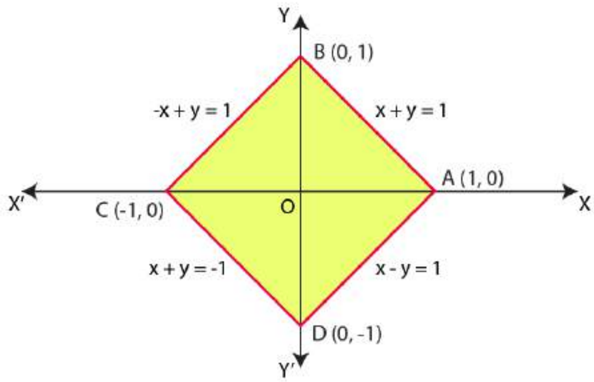
$= \frac{15}{2} - 3 = \frac{9}{2}$ sq. units

11. Using the method of integration, find the area enclosed by the curve $|x| + |y| = 1$.

[Hint: The required region is bounded by lines x + y = 1, x– y = 1, – x + y = 1 and – x – y = 1].

Solution: Equation of the curve is

$|x| + |y| = 1$ …(i)



The area bounded by the curve (i) is represented by the shaded region ABCD.

The curve intersects the axes at points A (1, 0), B (0, 1), C(-1, 0) and D(0, -1)

As, given curve is symmetrical about x-axis and y-axis.

Area bounded by the curve = Area of square ABCD = 4 x Δ OAB

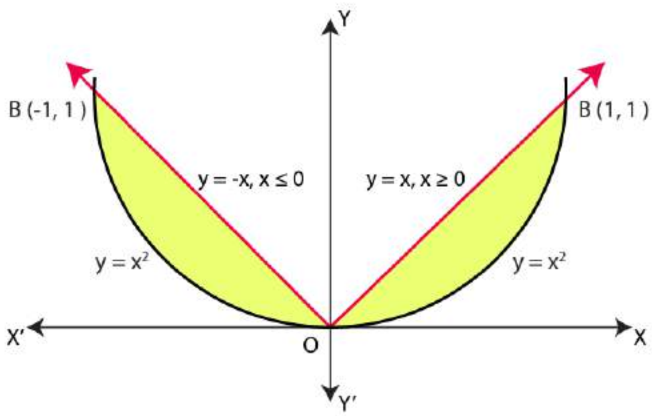
= $4\int_0^1 (1-x)\,dx$

= $4\left(x-\frac{x^2}{2}\right)_0^1$

= $4\times\frac{1}{2}$ = 2 sq. units

12. Find the area bounded by the curves $\{(x,y): y \geq x^2 \text{ and } y = |x|\}$**.**

Solution: The area bounded by the curves $\{(x,y): y \geq x^2 \text{ and } y = |x|\}$ is represented by the shaded region.



Since, area is symmetrical about y-axis.

Therefore, Required area = Area between parabola and x-axis between limits x=0 and x=1

$= \int_0^1 y\, dx = \int_0^1 x^2\, dx$

$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$ ....(i)

And Area of ray y=x and x-axis,

$= \int_0^1 y\, dx = \int_0^1 x\, dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$ ....(ii)

Required shaded area in first quadrant

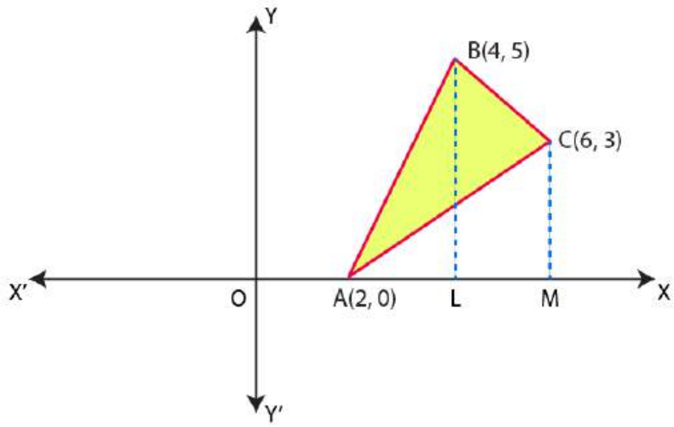
= (Area between ray $y = x$ for $x \geq 0$ and x-axis) – (Area between parabola $y = x^2$ and x-axis in first quadrant)

= Area given by equation (ii) – Area given by equation (i)

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

13.Using the method of integration, find the area of the triangle whose vertices are A (2, 0), B (4, 5) and C (6, 3).

Solution: Vertices of the given triangle are A (2, 0), B (4, 5) and C (6, 3).



Equation of side AB is $y-0=\frac{5-0}{4-2}(x-2)$

$= y=\frac{5}{2}(x-2)$

Equation of side BC is $y-5=\frac{3-5}{6-4}(x-4)$

$= y=9-x$

Equation of side AC is $y-0=\frac{3-0}{6-2}(x-2)$

$= y=\frac{3}{4}(x-2)$

Now, Required shaded area = Area ΔALB + Area of trapezium BLMC – Area ΔAMC

$= \int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (9-x)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$

$= \left[\frac{5}{2}(8-8)-(2-4)\right]+\left|54-18-(36-8)\right|-\left[\frac{3}{4}\{18-12-(2-4)\}\right]$

$= \frac{5}{2}(0+2)+\left|36-36+8\right|-\frac{3}{4}(6+2)$

= 5 + 8 – 6 = 7 sq. units

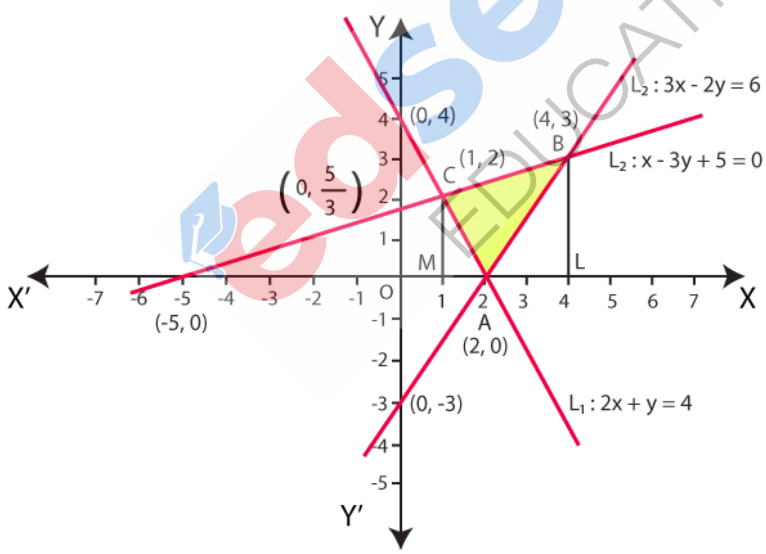
14. Using the method of integration, find the area of the region bounded by the lines: $2x+y=4$, $3x-2y=6$ **and** $x-3y+5=0$.

Solution:

Lets say, equation of one line l_1 is $2x+y=4$,
equation of second line l_2 is $3x-2y=6$

And equation of third line l_3 is $x-3y+5=0$.

Draw all the lines on the coordinate plane, we get



Here, vertices of triangle ABC are A (2, 0), B (4, 3) and C (1, 2).

Now, Required area of triangle = Area of trapezium CLMB – Area ΔACM – Area ΔABL

$= \int_1^4 \frac{1}{3}(x+5)\,dx - \int_1^2 (4-2x)\,dx - \int_2^4 \frac{3}{2}(x-2)\,dx$

$= \frac{1}{3}\left[8+20-\left(\frac{1}{2}+5\right)\right]-\{(8-4)-(4-1)\}-\frac{3}{2}|(8-8)-(2-4)|$

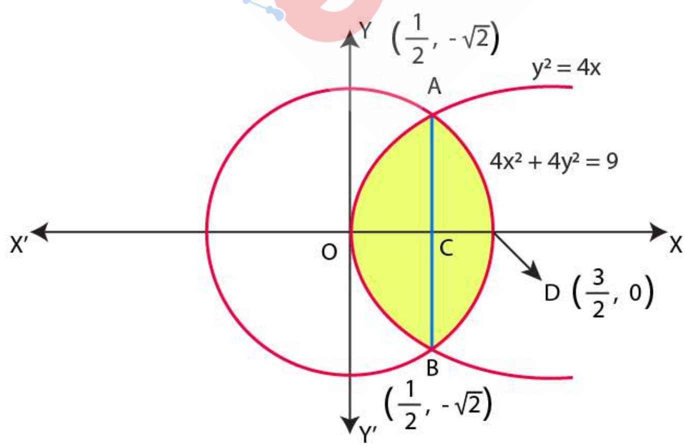
$= \frac{1}{3}\left(28-\frac{11}{2}\right)-(4-3)-\frac{3}{2}\times 2$

$= \frac{1}{3}\times\frac{45}{2}-1-3$

$= \frac{15}{2}-1-3=\frac{7}{2}$ sq. units

15.Find the area of the region $\{(x,y): y^2 \le 4x \text{ and } 4x^2+4y^2 \le 9\}$.

Solution: Equation of parabola is $y^2 = 4x$ ……(i)
And equation of circle is $4x^2+4y^2=9$ ……(ii)



From figures, points of intersection of parabola (i) and circle (ii) are

A $\left(\frac{1}{2}, \sqrt{2}\right)$ and B $\left(\frac{1}{2}, -\sqrt{2}\right)$

Required shaded area OADBO (Area of the circle which is interior to the parabola)

= 2 x Area OADO = 2 [Area OAC + Area CAD]

$$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\, dx\right]$$

$$= \left[\left\{2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{\frac{9}{4} x^2}}{2} + \frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{3/2}\right\}_{\frac{1}{2}}^{\frac{3}{2}}\right]$$

$$= 2\left[\frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8}\sin^{-1}1 - \frac{\frac{1}{2}\sqrt{2}}{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9}{8}.\frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \left(\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} + \frac{\sqrt{2}}{6}\right)$$ sq. units

16. Choose the correct answer:

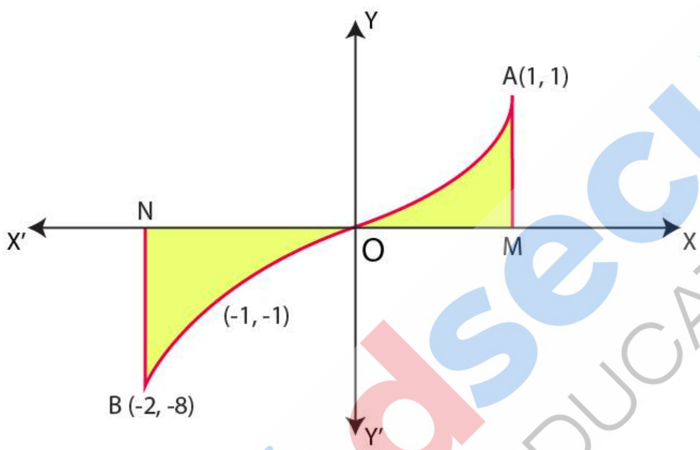
Area bounded by the curve $y=x^3$ the x-axis and the ordinate x=-2 and x=1 is:

(A) -9 (B) -15/4 (C) 15/4 (D) 17/4

Solution:
Option (D) is correct.

Explanation:

Equation of the curve is $y = x^3$



Now, Area OBN ($y = x^3$ for $-2 \le x \le 0$) and Area OAM ($y = x^3$ for $0 \le x \le 1$)

Therefore, Required area = Area OBN + Area OAM

$$= \int_{-2}^{0} x^3\, dx + \int_{0}^{1} x^3\, dx$$

$= \frac{17}{4}$ sq. units

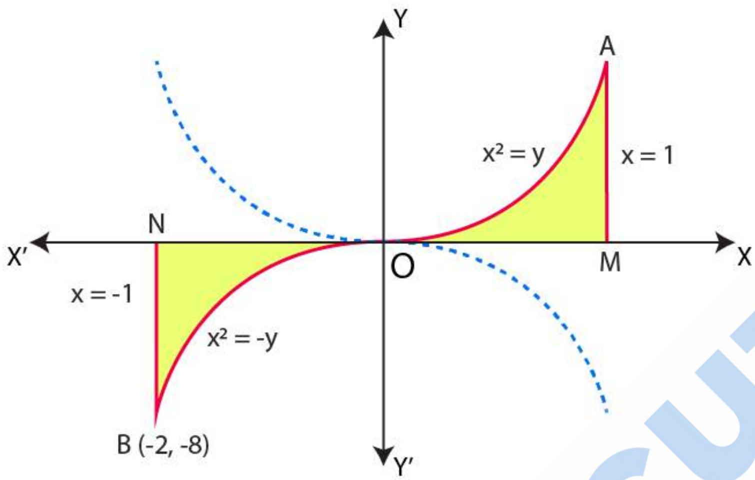
17. Choose the correct answer:

The area bounded by the curve $y = x|x|$, x- axis and the ordinates x = -1 and x = 1 is given by:
(A) 0 (B) 1/3 (C) 2/3 (D) 4/3

Solution:
Option (C) is correct.

Explanation:

Equation of the curve is



$y = x|x| = x(x) = x^2$ if $x \geq 0$ ....(1)

And $y = x|x| = x(-x) = -x^2$ if $x \leq 0$ ....(2)

Required area = Area ONBO + Area OAMO

$= \int_{-1}^{0} -x^2\, dx + \int_{0}^{1} x^2\, dx$

= 2/3 sq. units

18. Choose the correct answer:

The area of the circle $x^2 + y^2 = 16$ **exterior to the parabola** $y^2 = 6x$.

(A) $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$ **(B)** $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

(C) $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$ **(D)** $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

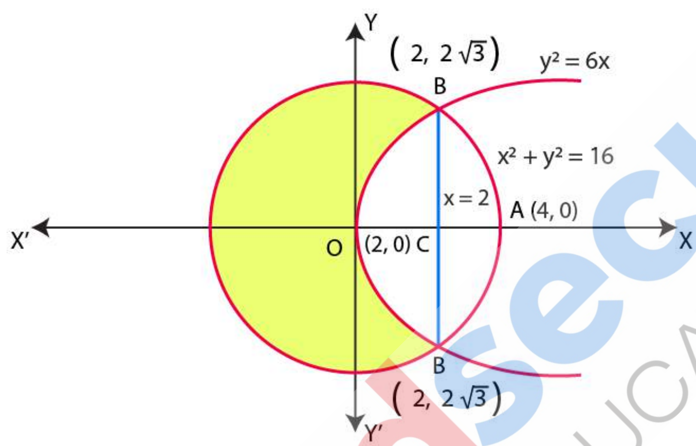
Solution:

Option (C) is correct.

Explanation:

Equation of the circle is $x^2 + y^2 = 16$ …..(1)
Thus, radius of circle is 4
This circle is symmetrical about x-axis and y- axis.

Here two points of intersection are B $\left(2, 2\sqrt{3}\right)$ and B' $\left(2, -2\sqrt{3}\right)$.



Required area = Area of circle – Area of circle interior to the parabola

= πr^2 - Area OBAB'O

= 16π - 2 x Area OBACO

= 16π - 2[Area OBCO + Area BACB]

= $16\pi - 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$

$= 16\pi - 2\left[\frac{2}{3}\sqrt{6}\left(2\sqrt{2}\right) + 8\sin^{-1}1 - \sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]$

$= 16\pi - 2\left[\frac{8}{\sqrt{3}} + 8.\frac{\pi}{2} - 2\sqrt{3} - 8.\frac{\pi}{6}\right]$

$= 16\pi - 2\left[\frac{8}{\sqrt{3}} - 2\sqrt{3} + 8\pi\left(\frac{1}{2} - \frac{1}{6}\right)\right]$

$= 16\pi - 2\left[\frac{2}{\sqrt{3}} + \frac{8\pi}{3}\right]$

$= 16\pi\left(1 - \frac{1}{3}\right) - \frac{4}{\sqrt{3}}$

$= \frac{4}{3}\left(8\pi - \sqrt{3}\right)$ sq. units

19. Choose the correct answer:

The area bounded by the y-axis, y = cos x and y= sin x when $0 \le x \le \frac{\pi}{2}$ is:

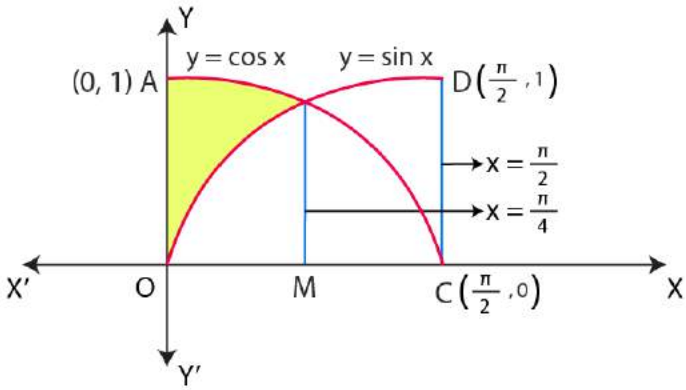
(A) $2\left(\sqrt{2} - 1\right)$ **(B)** $\sqrt{2} - 1$ **(C)** $\sqrt{2} + 1$ **(D)** $\sqrt{2}$

Solution:
Option (B) is correct.

Explanation:

Graph of both the functions are intersect at the point

B $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.



Required Shaded Area = Area OABC – Area OBC

= Area OABC – (Area OBM + Area BCM)

$$= \int_0^{\pi/2} \cos x \, dx - \left(\int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx \right)$$

$$= \left(\sin\frac{\pi}{2} - \sin 0° \right) - \left(-\cos\frac{\pi}{4} + \cos 0° + \sin\frac{\pi}{2} - \sin\frac{\pi}{4} \right)$$

$$= 1 + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} = \left(\sqrt{2} - 1\right)$$ sq. units