Exercise 5.1

1. Prove that the function f(x) = 5x - 3 is continuous at x = 0 at x = -3 and at x = 5.

Solution:
Given function is f(x) = 5x – 3

Continuity at x = 0,
$\lim_{x\to 0} f(x) = \lim_{x\to 0}(5x-3)$
= 5 (0) – 3
= 0 – 3
= – 3

Again, f(0) = 5 (0) – 3 = 0 – 3 = – 3

As $\lim_{x\to 0} f(x) = f(x)$, therefore, f(x) is continuous at x = 0.

Continuity at x = -3,

$\lim_{x\to -3} f(x) = \lim_{x\to -3}(5x-3)$ = 5 (– 3) – 3 = – 18

And f(-3)= 5 (– 3) – 3 = – 18

As $\lim_{x\to -3} f(x) = f(x)$, therefore, is continuous at x = -3

Continuity at x = 5,
$\lim_{x\to 5} f(x) = \lim_{x\to 5}(5x-3)$
= 5 (5) – 3 = 22

And f(5) = 5 (5) – 3 = 22

Therefore, $\lim_{x\to 5} f(x) = f(x)$, so, f(x) is continuous at x = -5.

2. Examine the continuity of the function f(x) = $2x^2$ – 1 at x = 3.

Solution:
Given function f(x) = $2x^2$ – 1

Check Continuity at x = 3,

$\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x^2 - 1)$

= 2(3)^2 – 1 = 17

And f(3) = 2(3)^2 – 1 = 17

Therefore, $\lim_{x\to 3} f(x) = f(x)$ so f(x) is continuous at x = 3.

3. Examine the following functions for continuity:

(a) $f(x) = x - 5$

(b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

(d) $f(x) = |x - 5|$

Solution:

(a) Given function is $f(x) = x - 5$

We know that, f is defined at every real number k and its value at k is $k - 5$.

Also observed that $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k - y = f(k)$

As, $\lim_{x\to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every real number and it is a continuous function.

(b) Given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we have

$\lim_{x\to k} f(x) = \lim_{x\to k} \frac{1}{x-5} = \frac{1}{k-5}$

and $f(k) = \frac{1}{k-5}$

As, $\lim_{x\to k} f(x) = f(k)$

Therefore,

f(x) is continuous at every point of domain of f and it is a continuous function.

(c) Given function is $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

For any real number, $k \neq -5$, we get

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{x^2 - 25}{x+5} = \lim_{x \to k} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to k}(x-5) = k-5$$

And $f(k) = \frac{(k+5)(k-5)}{k+5} = k-5$

As, $\lim_{x \to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every point of domain of f and it is a continuous function.

(d) Given function is $f(x) = |x-5|$

Domain of $f(x)$ is real and infinite for all real x

Here $f(x) = |x-5|$ is a modulus function.

As, every modulus function is continuous.

Therefore, f is continuous in its domain R.

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is a positive integer.

Solution: Given function is $f(x) = x^n$ where n is a positive integer.

Continuity at $x = n$, $\lim_{x \to n} f(x) = \lim_{x \to n}(x^n) = n^n$

And $f(n) = n^n$

As, $\lim_{x \to n} f(x) = f(x)$, therefore, $f(x)$ is continuous at $x = n$.

5. Is the function f defined by $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$ **continuous at** $x=0$, **at** $x=1$, **at** $x=2$?

Solution: Given function is $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$

Step 1: At $x=0$, We know that, f is defined at 0 and its value 0.

Then $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x=0$ and $f(0)=0$

Therefore, $f(x)$ is continuous at $x=0$.

Step 2: At $x=1$, Left Hand limit (LHL) of f $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x)=1$

Right Hand limit (RHL) of f $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x)=5$

Here $\lim\limits_{x\to 1^-} f(x)\ne\lim\limits_{x\to 1^+} f(x)$

Therefore, $f(x)$ is not continuous at $x=1$.

Step 3: At $x=2$, f is defined at 2 and its value at 2 is 5.

$\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}(5)=5$, therefore, $\lim\limits_{x\to 2} f(x)=f(2)$

Therefore, $f(x)$ is not continuous at $x=2$.

Find all points of discontinuity of f, where f is defined by:

6. $f(x)=\begin{cases} 2x+3, & x\le 2 \\ 2x-3, & x>2 \end{cases}$

Solution: Given function is $f(x)=\begin{cases} 2x+3, & \text{if } x\le 2 \\ 2x-3, & \text{if } x>2 \end{cases}$

Here $f(x)$ is defined for $x\le 2$ or $(-\infty,2)$ and also for $x>2$ or $(2,\infty)$.

Therefore, Domain of f is $(-\infty,2)\cup(2,\infty)=(-\infty,\infty)$ = R

Therefore, For all $x<2, f(x)=2x+3$ is a polynomial and hence continuous and for all $x>2, f(x)=2x-3$ is a continuous and hence it is also continuous on R – {2}.

Now Left Hand limit = $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)$ = 2 x 2 + 3 = 7

Right Hand limit = $\lim_{x\to 2+} f(x)=\lim_{x\to 2^-}(2x-3)$ = 2 x 2 – 3 = 1

As, $\lim_{x\to 2^-} f(x)\neq \lim_{x\to 2^-} f(x)$

Therefore, $\lim_{x\to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at only x = 2.

Find all points of discontinuity of f, where f is defined by:

7. $$f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$$

$$f(x)=\begin{cases}|x|+3, & \text{if } x\le 2\\ -2x, & \text{if } x>2\\ 6x+2, & \text{if } x\ge 3\end{cases}$$

Solution: Given function is

Here $f(x)$ is defined for $x\le -3$ or $(-\infty,-3)$ and for $-3<x<3$ and also for $x\ge 3$ or $(3,\infty)$.

Therfore, Domain of f is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)=(-\infty,\infty)$ = R

Therfore, For all $x<-3, f(x)=|x|+3=-x+3$ is a polynomial and hence continuous and

for all $x(-3<x<3), f(x)=-2x$ is a continuous and a continuous function and also

for all $x>3, f(x)=6x+2$.

Therefore, f(x) is continuous on R – {-3, 3}.

And, x = -3 and x = 3 are partitioning points of domain R.

Now, Left Hand limit = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (|x|+3) = \lim_{x\to 3^-} (-x+3) = 3+3 = 6$

Right Hand limit = $\lim_{x\to 3+} f(x) = \lim_{x\to 3^-} (-2x) = (-2)(-3) = 6$

And $f(-3) = |-3|+3 = 3+3 = 6$

Therefore, $f(x)$ is continuous at $x=-3$.

Again,Left Hand limit = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x) = -2(3) = -6$

Right Hand limit = $\lim_{x\to 3+} f(x) = \lim_{x\to 3^-} (6x+2) = 6(3)+2 = 20$

As, $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^-} f(x)$

Therefore, $\lim_{x\to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at only x = 3.

Find all points of discontinuity of f, where f is defined by:

8.

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Solution: Given function is $f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

f(x) = |x|/x can also be defined as,

$\frac{x}{x} = 1$ if $x > 0$ and $\frac{-x}{x} = -1$ if $x < 0$

$\Rightarrow f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$ and $f(x) = 0$ if $x = 0$

We get that, domain of $f(x)$ is R as $f(x)$ is defined for $x > 0$, $x < 0$ and $x = 0$.

For all $x>0$, $f(x)=1$ is a constant function and continuous.

For all $x<0$, $f(x)=-1$ is a constant function and continuous.

Therefore $f(x)$ is continuous on R – {0}.

Now,

Left Hand limit = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-1) = -1$

Right Hand limit = $\lim_{x\to 0+} f(x) = \lim_{x\to 0^-}(1) = 1$

As, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^-} f(x)$

Therefore, $\lim_{x\to 0} f(x)$ does not exist and f(x) is discontinuous at only x = 0.

Find all points of discontinuity of f, where f is defined by:

9.

$$f(x)=\begin{cases}\frac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\geq 0\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}\frac{x}{|x|}, & \text{if} \quad x<0\\ -1, & \text{if} \quad x\geq 0\end{cases}$$

At $x=0$, L.H.L. = $\lim_{x\to 0^-}\frac{x}{|x|}=-1$ And $f(0)=-1$

R.H.L. = $\lim_{x\to 0^-} f(x) = -1$

As, L.H.L. = R.H.L. = $f(0)$

Therefore, $f(x)$ is a continuous function.

Now,

for $x = c < 0$ $\lim_{x \to c^-} \frac{x}{|x|} = -1 = f(c)$

Therefore, $\lim_{x \to c^-} = f(x)$

Therefore, $f(x)$ is a continuous at $x = c < 0$

Now, for $x = c > 0$ $\lim_{x \to c^-} f(x) = 1 = f(c)$

Therefore, $f(x)$ is a continuous at $x = c > 0$

Answer: The function is continuous at all points of its domain.

Find all points of discontinuity of f, where f is defined by:
10.
$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Solution: Given function is
$$f(x) = \begin{cases} x+1, & \text{if} \quad x \geq 1 \\ x^2+1, & \text{if} \quad x < 1 \end{cases}$$

We know that, f(x) being polynomial is continuous for $x \geq 1$ and $x < 1$ for all $x \in$ R.

Check Continuity at x = 1

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = \lim_{h \to 0} (1+h+1) = 2$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = \lim_{h \to 0} \left((1-h)^2 + 1\right) = 2$

And $f(1) = 2$

As, L.H.L. = R.H.L. = $f(1)$

Therefore, $f(x)$ is a continuous at $x = 1$ for all $x \in$ R.

Hence, $f(x)$ has no point of discontinuity.

Find all points of discontinuity of f, where f is defined by:
11.

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}x^3-3, & \text{if} \quad x\le 2\\ x^2+1, & \text{if} \quad x>2\end{cases}$$

At $x=2$, L.H.L. = $\lim\limits_{x\to 2^-}(x^3-3)=8-3=5$

R.H.L. = $\lim\limits_{x\to 2^+}(x^2+1)=4+1=5$

$f(2)=2^3-3=8-3=5$

As, L.H.L. = R.H.L. = $f(2)$

Therefore, $f(x)$ is a continuous at $x=2$

Now, for $x=c<0$ $\lim\limits_{x\to c}(x^3-3)=c^3-3=f(c)$ and

$\lim\limits_{x\to c}(x^2+1)=c^2+1=f(c)$

Therefore, $\lim\limits_{x\to c^-}=f(x)$

This implies, $f(x)$ is a continuous for all $x\in$ R.

Hence the function has no point of discontinuity.

Find all points of discontinuity of f, where f is defined by:

12. $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$

Solution: Given function is

$$f(x)=\begin{cases} x^{10}-1, & \text{if } \quad x\le 1 \\ x^2, & \text{if } \quad x>1 \end{cases}$$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=0$

R.H.L. = $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1$

$f(1)=1^{10}-1=0$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$ $\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1=f(c)$ and for $x=c>1$ $\lim_{x\to c}\left(x^2\right)=c^2=f(1)$

Therefore, $f(x)$ is a continuous for all $x\in$ R – {1}

Hence for all given function $x=1$ is a point of discontinuity.

13. Is the function defined by $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>11 \end{cases}$ **a continuous function?**

Solution: Given function is $f(x)=\begin{cases} x+5, & \text{if } \quad x\le 1 \\ x-5, & \text{if } \quad x>1 \end{cases}$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=6$

R.H.L. = $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=-4$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$

$\lim_{x\to c}(x+5)=c+5=f(c)$ and

for $x=c>1$ $\lim_{x\to c}(x-5)=c-5=f(c)$

Therefore, f(x) is a continuous for all x ∈ R – {1}

Hence f(x) is not a continuous function.

14. **Discuss the continuity of the function** f, **where** f **is defined by:**

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

In interval, $0\le x\le 1,$ $f(x)=3$

Therefore, f is continuous in this interval.

At x =1,

L.H.L. = $\lim_{x\to 1^-} f(x)=3$ and R.H.L. = $\lim_{x\to 1^+} f(x)=4$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at x = 1.

At $x=3,$ L.H.L. = $\lim_{x\to 3^-} f(x)=4$ and R.H.L. = $\lim_{x\to 3^+} f(x)=5$

As, L.H.L. $\neq$ R.H.L.

Therefore, f(x) is discontinuous at x = 3

Hence, f is discontinuous at x = 1 and x = 3.

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$$

15.

Solution: Given function is

$$f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$$

At x =0, L.H.L. = $\lim\limits_{x\to 0^-} 2x=0$ and R.H.L. = $\lim\limits_{x\to 0^+}(0)=0$

As, L.H.L. = R.H.L.

Therefore, f(x) is continuous at x = 0

At x = 1, L.H.L. = $\lim\limits_{x\to 1^-}(0)=0$ and R.H.L. = $\lim\limits_{x\to 1^+}(4x)=4$

As, L.H.L. ≠ R.H.L.

Therefore, f(x) is discontinuous at x = 1.

When x<0,

f(x) is a polynomial function and is continuous for all x < 0.

When $x>1, f(x)=4x$

It is being a polynomial function is continuous for all $x>1$.

Hence, x = 1 is a point of discontinuity.

Discuss the continuity of the function f, where f is defined by

16. $f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$

Solution: Given function is

$$f(x)=\begin{cases}-2, & \text{if} \quad x\le -1\\ 2x, & \text{if} \quad -1<x\le 1\\ 2, & \text{if} \quad x>1\end{cases}$$

At x = -1,

L.H.L. = $\lim_{x\to -1^-} f(x)=-2$ and R.H.L. = $\lim_{x-10^+} f(x)=-2$

As, L.H.L. = R.H.L.

Therefore, $f(x)$ is continuous at $x=-1$

At x = 1,

L.H.L. = $\lim_{x\to 1^-} f(x)=2$ and R.H.L. = $\lim_{x\to 1^+} f(x)=2$

As, L.H.L. = R.H.L.

Therefore, f(x) is continuous at x = 1.

17. Find the relationship between a and b so that the function f defined by $f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+1, & \text{if } x>3\end{cases}$ **is continuous at x = 3**

Solution: Given function is

$$f(x)=\begin{cases}ax+1, & \text{if} \quad x\le 3\\ bx+3, & \text{if} \quad x>3\end{cases}$$

Check Continuity at $x=3,$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(ax+1) = \lim_{h\to 0}\{a(3-h)+1\} = \lim_{h\to 0}(3a-ah+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(bx+3) = \lim_{h\to 0}\{b(3+h)+3\} = \lim_{h\to 0}(3b+bh+3) = 3b+3$$

Also $f(3)=3a+1$

Therefore, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \; 3b+3=3a+1$

$\Rightarrow \; a-b=\frac{2}{3}$

18. For what value of λ is the function defined by

$$f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Solution: Since $f(x)$ is continuous at $x=0.$
Therefore,

L.H.L.

$$\lim_{x\to 0^-} f(x) = f(0) = \lambda(x^2-2x) = \lambda(0-0) = 0$$

R.H.L

And $\lim_{x\to 0^+} f(x) = f(0) = 4x+1 = 4\times 0+1 = 1$

Here, L.H.L. ≠ R.H.L.

This implies 0 = 1, which is not possible.

Again, f(x) is continuous at x = 1.

Therefore,

$$\lim_{x\to 1^-} f(x) = f(-1) = \lambda(x^2 - 2x) = \lambda(1+2) = 3\lambda$$

And $\lim_{x\to 1^+} f(x) = f(1) = 4x+1 = 4\times 1+1 = 5$

Let us say, L.H.L. = R.H.L.

$\Rightarrow 3\lambda = 5$

$\Rightarrow \lambda = \frac{5}{3}$

The value of is 3/5.

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution: For any real number, x,

$[x]$ denotes the fractional part or decimal part of x.

For example,

[2.35] = 0.35

[–5.45] = 0.45

[2] = 0

[–5] = 0

The function g : R -> R defined by $g(x) = x - [x] \forall x \in \infty$ is called the fractional part function.

The domain of the fractional part function is the set R of all real numbers , and

[0, 1) is the range of the set.

So, given function is discontinuous function.

20. Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution: Given function is $f(x) = x^2 - \sin x + 5$

L.H.L. = $\lim_{x \to \pi^-} \left(x^2 - \sin x + 5\right) = \lim_{x \to \pi^-} \left[(\pi - h)^2 - \sin(\pi - h) + 5\right] = \pi^2 + 5$

R.H.L. = $\lim_{x \to \pi^+} \left(x^2 - \sin x + 5\right) = \lim_{x \to \pi^-} \left[(\pi + h)^2 - \sin(\pi + h) + 5\right] = \pi^2 + 5$

And $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 + 5$

Since L.H.L. = R.H.L. = $f(\pi)$

Therefore, f is continuous at $x = \pi$

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x . \cos x$

Solution: **(a)** Let "a" be an arbitrary real number then

$\lim_{x \to a^+} f(x) \Rightarrow \lim_{h \to 0} f(a + h)$

Now,

$\lim_{h \to 0} f(a + h) = \lim_{h \to 0} \sin(a + h) + \cos(a + h)$

= $\lim_{h \to 0} (\sin a \cos h + \cos a \sin h + \cos a \cos h - \sin a \sin h)$

= $\sin a \cos 0 + \cos a \sin 0 + \cos a \cos 0 - \sin a \sin 0$

{As cos 0 = 1 and sin 0 = 0}

= $\sin a + \cos a = f(a)$

Similarly,

$\lim_{x\to a^-} f(x) = f(a)$

$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^-} f(x)$

Therefore, f(x) is continuous at x = a.

As, “a” is an arbitrary real number, therefore, $f(x) = \sin x + \cos x$ is continuous.

(b) Let “a” be an arbitrary real number then $\lim_{x\to a^-} f(x) \Rightarrow \lim_{h\to 0} f(a+h)$
Now,

$\lim_{h\to 0} f(a+h) =$ $\lim_{h\to 0} \sin(a+h) - \cos(a-h)$

$\Rightarrow$ $\lim_{h\to 0}(\sin a\cos h + \cos a\sin h - \cos a\cos h - \sin a\sin h)$

= $\sin a\cos 0 + \cos a\sin 0 - \cos a\cos 0 - \sin a\sin 0$

= $\sin a + 0 - \cos a - 0$

= $\sin a - \cos a = f(a)$

Similarly, $\lim_{x\to a^-} f(x) = f(a)$

$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^-} f(x)$

Therefore, f(x) is continuous at x = a.

Since, “a” is an arbitrary real number, therefore, $f(x) = \sin x - \cos x$ is continuous.

(c) Let “a” be an arbitrary real number then $\lim_{x\to a^-} f(x) \Rightarrow \lim_{h\to 0} f(a+h)$

$Now, \lim_{h\to 0} f(a+h) =$ $\lim_{h\to 0} \sin(a+h).\cos(a+h)$

= $\lim_{h\to 0}(\sin a\cos h + \cos a\sin h)(\cos a\cos h - \sin a\sin h)$

= $(\sin a\cos 0+\cos a\sin 0)(\cos a\cos 0-\sin a\sin 0)$

= $(\sin a+0)(\cos a-0)$

= $\sin a.\cos a=f(a)$

Similarly, $\lim_{x\to a^-} f(x)=f(a)$

$\lim_{x\to a^-} f(x)=f(a)=\lim_{x\to a^-} f(x)$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $f(x)=\sin x.\cos x$ is continuous.

22. Discuss the continuity of cosine, cosecant, secant and cotangent functions.

Solution:

Continuity of cosine:
Let say "a" be an arbitrary real number then
$\lim_{x\to a^-} f(x)\Rightarrow \lim_{x\to a^-}\cos x\Rightarrow \lim_{h\to 0}\cos(a+h)$

Which implies, $\lim_{h\to 0}(\cos a\cos h-\sin a\sin h)$

= $\cos a\lim_{h\to 0}\cos h-\sin a\lim_{h\to 0}\sin h$

= $\cos a\times 1-\sin a\times 0$ = $\cos a=f(a)$

$\lim_{x\to a} f(x)=f(a)$ for all $a\in$ R

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $\cos x$ is continuous.

Continuity of cosecant:

Let say "a" be an arbitrary real number then

$f(x) = \cos ec\ x = \frac{1}{\sin x}$ and

domain $x = \mathrm{R} - (x\pi), x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\sin x} = \frac{1}{\lim_{h \to 0} \sin(a+h)}$

$= \frac{1}{\lim_{h \to 0}(\sin a \cos\ h + \cos a \sin\ h)}$

$= \frac{1}{\sin a \cos 0 + \cos a \sin 0}$

$= \frac{1}{\sin a(1) + \cos a(0)}$

$= \frac{1}{\sin a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, “a” is an arbitrary real number, therefore, $f(x) = \cos ec\ x$ is continuous.

Continuity of secant:

Let say “a” be an arbitrary real number then

$f(x) = \sec\ x = \frac{1}{\cos x}$ and domain $x = \mathrm{R} - (2x+1)\frac{\pi}{2}, x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\cos x} = \frac{1}{\lim_{h \to 0} \cos(a+h)}$

$= \frac{1}{\lim_{h \to 0}(\cos a \cos\ h - \sin a \sin\ h)}$

$= \frac{1}{\cos a \cos 0 - \sin a \sin 0}$

$= \frac{1}{\cos a(1) - \sin a(0)} = \frac{1}{\cos a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $f(x) = \sec x$ is continuous.

Continuity of cotangent:

Let say "a" be an arbitrary real number then

$f(x) = \cot x = \frac{1}{\tan x}$ and domain $x = \mathrm{R} - (x\pi), x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\tan x} = \frac{1}{\lim_{h \to 0} \tan(a+h)}$

$= \frac{1}{\lim_{h \to 0}\left(\frac{\tan a + \tan h}{1 - \tan a \tan h}\right)} = \frac{1}{\frac{\tan a + 0}{1 - \tan a \tan 0}}$

$= \frac{1-0}{\tan a} = \frac{1}{\tan a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $f(x) = \cot x$ is continuous.

23. Find all points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$**.**

Solution: Given function is

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

At x = 0,

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(-h)}{-h} = 1$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 0+1 = 1$

$f(0) = 1$

Therefore, f is continuous at $x = 0.$

When $x < 0, \sin x$ and x are continuous, then $\frac{\sin x}{x}$ is also continuous.

When $x > 0, f(x) = x+1$ is a polynomial, then f is continuous.

Therefore, f is continuous at any point.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function.

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

As we know, sin(1/x) lies between -1 and 1, so the value of sin 1/x be any integer, say m, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$
= 0 x m
= 0
And, f(0) = 0

Since, $\lim_{x \to 0} f(x) = f(0),$ therefore, the function f is continuous at $x = 0.$

25. Examine the continuity of f, where f is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}.$$

Solution:

Given function is

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$$

Find Left hand and right land limits at x =0.

At x = 0, L.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0-h)=\lim_{h\to 0} f(-h)$

$\Rightarrow \lim_{h\to 0}\sin(-h)+\cos(-h) = \lim_{h\to 0}(-\sin h+\cos h)=-0+1=1$

R.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0+h)=\lim_{h\to 0} f(h)$

$\Rightarrow \lim_{h\to 0}\sin(h)+\cos(h) = \lim_{h\to 0}(\sin h+\cos h)=0+1=1$

And f(0) = 1

Therefore, $\lim_{h\to 0^-} f(x)=\lim_{h\to 0^-} f(x)\neq f(0)$

Therefore, f(x) is discontinuous at x = 0.

Find the values of k so that the function f is continuous at the indicated point in Exercise 26 to 29.

26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ **at** $x=\dfrac{\pi}{2}$.

Solution:

Given function is

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

So, $x\to\dfrac{\pi}{2}$

This implies, $x\neq\dfrac{\pi}{2}$

Putting $x=\dfrac{\pi}{2}+h$ where $h\to 0$

$$=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=\lim_{h\to 0}\frac{-k\sin h}{\pi-\pi-2h}$$

$$=\lim_{h\to 0}\frac{-k\sin h}{-2h}$$

$$=\frac{k}{2}\times\lim_{h\to 0}\frac{\sin h}{h}$$

$=\dfrac{k}{2}$ ....(1)

And $f\left(\dfrac{\pi}{2}\right)=3$ ....(2)

$f(x)=3$ when $x=\dfrac{\pi}{2}$ [Given]

As we know, f(x) is continuous at x= π/2.

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

From equation (1) and equation (2), we have

$$\frac{k}{2} = 3$$

k = 6

Therefore, the value of k is 6.

27. $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$ **at** $x=2$.

Solution:
Given function is

$$f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2+h) = 3$$

$\lim_{x\to 2^-} f(x) = 3$ and $f(2)=3$

$$k\times 2^2 = 3$$

This implies, $k=\frac{3}{4}$

when k=3/4, then $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{3}{4}(2-h)^2 = 3$

Therefore, $f(x)$ is continuous at $x=2$ when $k=\frac{3}{4}$.

28. $f(x)=\begin{cases} kx+1, & \text{if } x\le \pi \\ \cos x, & \text{if } x>\pi \end{cases}$ **at** $x=\pi$.

Solution:
Given function is:

$$f(x)=\begin{cases} kx+1, & \text{if } x\le \pi \\ \cos x, & \text{if } x>\pi \end{cases}$$

$$\lim_{x\to\pi^-} f(x)=\lim_{h\to 0} f(\pi+h)=\lim_{h\to 0}\cos(\pi+h)=-\cos h=-\cos 0=-1$$

and $\lim_{x\to\pi^-} f(x)=\lim_{h\to 0} f(\pi-x)=\lim_{h\to 0}\cos(\pi-h)=-\cos h=-\cos 0=-1$

Again,

$$\lim_{x\to\pi} f(x)=\lim_{h\to 0}(k\pi+1)$$

As given function is continuous at $x=\pi$, we have

$$\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi} f(x)$$

$$\Rightarrow k\pi+1=-1$$

$$\Rightarrow k\pi=-2$$

$$\Rightarrow k=\frac{-2}{\pi}$$

The value of k is -2/π.

29. $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$ **at** $x=5$.

Solution:
Given function is

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$$

When x< 5, $f(x) = kx + 1$: A polynomial is continuous at each point x < 5.

When x > 5, $f(x) = 3x - 5$: A polynomial is continuous at each point x > 5.

Now $f(5) = 5k + 1 = 3(5 + h) - 5$

$\lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5+h) = 15 + 3h - 5$ ....(1)

$= 10 + 3h = 10 + 3 \times 0 = 10$

$\lim_{x \to 5^-} f(x) = \lim_{h \to 0} f(5-h) = k(5-h) + 1 = 5k - nk + 1 = 5k + 1$ ....(2)

Since function is continuous, therefore, both the equations are equal,

Equate both the equations and find the value of k,

$10 = 5k + 1$

$5k = 9$

$k = \frac{9}{5}$

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Solution:

Given function is:

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

For x < 2 ; function is f(x) =5 ; which is a constant.

Function is continuous.

For 2 < x < 10; function f(x) = ax + b; a polynomial.

Function is continuous.

For x > 10; function is f(x) =21; which is a constant.

Function is continuous.

Now, for continuity at $x=2,$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{h\to 0}(5) = \lim_{h\to 0}\{a(2+h)+b\} = 5$

$\Rightarrow 2a+b=5$ ....(1)

For continuity at $x=10,$ $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{h\to 0}(21) = \lim_{h\to 0}\{a(10-h)+b\} = 21$

$\Rightarrow 10a+b=21$ ....(2)

Solving equation (1) and equation (2), we get

a = 2 and b = 1.

31. Show that the function defined by f(x) = cos (x^2) is a continuous function.

Solution:
Given function is :
f(x) = cos (x^2)

Let g(x) = cos x and h(x) = x^2, then

goh(x) = g(h(x))

= g(x^2)

= cos (x^2)

=f(x)

This implies, goh(x) = f(x)
Now,
g(x) = cos x is continuous and
h(x) = x^2 (a polynomial)

[We know that, if two functions are continuous then their composition is also continuous]

So, goh(x) is also continuous.

Thus f(x) is continuous.

32. Show that the function defined by f(x) = |cos x| is a continuous function.

Solution: Given function is
f(x) = |cos x|

f(x) is a real and finite for all x ∊ R and Domain of f(x) is R.

Let $g(x) = \cos x$ and $h(x) = |x|$

Here, g(x) and h(x) are cosine function and modulus function are continuous for all real x.

Now, $(goh)x = g\{h(x)\} = g(|x|) = \cos|x|$ is also is continuous being a composite function of two continuous functions, but not equal to f(x).

Again, $(hog)x = h\{g(x)\} = h(\cos x) = |\cos x| = f(x)$ [Using given]

Therefore, $f(x) = |\cos x| = (hog)x$ is composite function of two continuous functions is continuous.

33. Examine that $\sin|x|$ is a continuous function.

Solution:
Let $f(x) = |x|$ and $g(x) = \sin|x|$, then
$(gof)x = g\{f(x)\} = g(|x|) = \sin|x|$

Now, f and g are continuous, so their composite, (gof) is also continuous.

Therefore, $\sin|x|$ is continuous.

34. Find all points of discontinuity of f defined by f(x) = |x| - |x +1|
Solution:

Given function is $f(x)=|x|-|x+1|$

When x < -1: $f(x)=-x-\{-(x+1)\}$ = $-x+x+1=1$

When $-1\le x<0$; $f(x)=-x-(x+1)=-2x-1$

When $x\ge 0,$; $f(x)=x-(x+1)=1$

So, we have a function as:

$$f(x)=\begin{cases} 1, & \text{if} \quad x<-1 \\ -2x-1, & \text{if} \quad -1\le x<0 \\ \text{-1}, & \text{if} \quad x\ge 0 \end{cases}$$

Check the continuity at x = -1, x = 0

At $x=-1,$ L.H.L. = $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 1 = 1$

R.H.L. = $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-2x-1) = 1$

And $f(-1)=-2\times 1-1=1$

Therefore, at $x=-1,$ $f(x)$ is continuous.

At $x=0,$ L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-2x-1) = -1$ and R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-1) = -1$

And $f(0)=-1$

Therefore, at x = 0, f(x) is continuous.

There is no point of discontinuity.

Exercise 5.2 Page No: 166

Differentiate the functions with respect to x in Exercise 1 to 8.

1. $\sin\left(x^2+5\right)$

Solution: Let $y=\sin\left(x^2+5\right)$

Apply derivative both the sides with respect to x.

$\frac{dy}{dx}= \cos\left(x^2+5\right)\frac{d}{dx}\left(x^2+5\right)$

$= \cos\left(x^2+5\right)\left(2x+0\right)$

$= 2x\cos\left(x^2+5\right)$

2. $\cos\left(\sin x\right)$

Solution: Let $y=\cos\left(\sin x\right)$
Apply derivative both the sides with respect to x.

$\frac{dy}{dx}= -\sin\left(\sin x\right)\frac{d}{dx}\sin x$

$= -\sin\left(\sin x\right)\cos x$

3. $\sin\left(ax+b\right)$

Solution: Let $y=\sin\left(ax+b\right)$

Apply derivative both the sides with respect to x.

$\frac{dy}{dx}= \cos\left(ax+b\right)\frac{d}{dx}\left(ax+b\right)$

$= \cos\left(ax+b\right)\left(a+0\right) = a\cos\left(ax+b\right)$

4. $\sec\left(\tan\sqrt{x}\right)$

Solution: Let $y=\sec\left(\tan\sqrt{x}\right)$

Apply derivative both the sides with respect to x.

$$\frac{dy}{dx}=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}\frac{d}{dx}\sqrt{x}$$

$$=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}.\frac{1}{2}x^{\frac{1}{2}-1}$$

$$=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}.\frac{1}{2\sqrt{x}}$$

5. $\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$

Solution: Let $y=\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$

Using quotient rule,

$$\frac{dy}{dx}=\frac{\cos\left(cx+d\right)\frac{d}{dx}\sin\left(ax+b\right)-\sin\left(ax+b\right)\frac{d}{dx}\cos\left(cx+d\right)}{\cos^2\left(cx+d\right)}$$

$$=\frac{\cos\left(cx+d\right)\cos\left(ax+b\right).\frac{d}{dx}\left(ax+b\right)-\sin\left(ax+b\right)\left\{-\sin\left(cx+d\right)\right\}\frac{d}{dx}\left(cx+d\right)}{\cos^2\left(cx+d\right)}$$

$$=\frac{\cos\left(cx+d\right)\cos\left(ax+b\right)\left(a\right)+\sin\left(ax+b\right)\sin\left(cx+d\right)\left(c\right)}{\cos^2\left(cx+d\right)}$$

6. $\cos x^3\sin^2\left(x^5\right)$

Solution: Let $y=\cos x^3.\sin^2\left(x^5\right)$

Apply derivative both the sides with respect to x.

$$\frac{dy}{dx}=\cos x^3\frac{d}{dx}\sin^2\left(x^5\right)+\sin^2\left(x^5\right)\frac{d}{dx}\cos x^3$$

$= \cos x^3 .2\sin(x^5)\frac{d}{dx}\sin(x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}x^3$

$= \cos x^3 .2\sin(x^5)\cos(x^5)(5x^4) - \sin^2(x^5)\sin x^3 .3x^2$

$= 10x^4\cos x^3\sin(x^5)\cos(x^5) - 3x^2\sin^2(x^5)\sin x^3$

7. $2\sqrt{\cot(x^2)}$

Solution: Let $y = 2\sqrt{\cos(x^2)}$

Apply derivative both the sides with respect to x.

$\frac{dy}{dx} = 2.\frac{1}{2}\{\cot(x^2)\}^{\frac{-1}{2}}.\frac{d}{dx}\cot(x^2)$

$= \frac{1}{\sqrt{\cot(x^2)}}.\{-\cos ec(x^2)\}\frac{d}{dx}x^2$

$= \frac{1}{\sqrt{\cot(x^2)}}.\{-\cos ec(x^2)\}(2x)$

$= \frac{-2x\cos ec(x^2)}{\sqrt{\cot(x^2)}}$

8. $\cos(\sqrt{x})$

Solution: Let $y = \cos(\sqrt{x})$

Apply derivative both the sides with respect to x.

$\frac{dy}{dx} = -\sin\sqrt{x}\frac{d}{dx}\sqrt{x}$

$= -\sin\sqrt{x}.\frac{1}{2}(x)^{\frac{-1}{2}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

9. Prove that the function f given by $f(x)=|x-1|, x\in \mathrm{R}$ is not differentiable at x = 1.

Solution: Given function: $f(x)=|x-1|$

$f(1)=|1-1|=0$

Right hand limit: $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

= $\lim_{h\to 0}\frac{|1+h-1|-0}{h}$

= $\lim_{h\to 0}\frac{|h|}{h}$ = $\lim_{h\to 0}\frac{h}{h}$ = 1

and Left hand limit:

$f'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

= $\lim_{h\to 0}\frac{|1-h-1|-0}{-h}$

= $\lim_{h\to 0}\frac{|-h|}{-h}$

= $\lim_{h\to 0}\frac{-h}{h}=-1$

Right hand limit ≠ Left hand limit

Therefore, f(x) is not differentiable at x = 1.

10. Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at x = 1 and x = 2

Solution: Given function is

$f(x)=[x], 0<x<3$

Right hand limit:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$= \lim_{h\to 0} \frac{|1+h| - 1}{h}$

$= \lim_{h\to 0} \frac{1-1}{h}$

$= \lim_{h\to 0} \frac{0}{h}$ = 0

and Left hand limit

$$f'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$= \lim_{h\to 0} \frac{|1-h| - 1}{-h}$

$= \lim_{h\to 0} \frac{0-1}{-h}$ = ∞

Right hand limit ≠ Left hand limit

Therefore, $f(x) = [x]$ is not differentiable at x = 1.

In same way, $f(x) = [x]$ is not differentiable at x = 2.

Exercise 5.3

Find $\frac{dy}{dx}$ in the following Exercise 1 to 15.

1. $2x+3y=\sin x$

Solution: Given function is $2x+3y=\sin x$
Derivate function with respect to x, we have

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}\sin x$$

$$2+3\frac{dy}{dx}=\cos x$$

$$3\frac{dy}{dx}=\cos x-2$$

$$\frac{dy}{dx}=\frac{\cos x-2}{3}$$

2. $2x+3y=\sin y$

Solution: Given function is $2x+3y=\sin y$
Derivate function with respect to x, we have

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}\sin y$$

$$2+3\frac{dy}{dx}=\cos y\frac{dy}{dx}$$

$$-\frac{dy}{dx}(\cos y-3)=-2$$

$$\frac{dy}{dx}=\frac{2}{\cos y-3}$$

3. $ax+by^2=\cos y$

Solution: Given function is $ax+by^2=\cos y$
Derivate function with respect to x, we have

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}\cos y$$

$$a+b.2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$2by\frac{dy}{dx}+\sin y\frac{dy}{dx}=-a$$

$$-\frac{dy}{dx}(2by+\sin y)=-a$$

$$\frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

4. $xy+y^2=\tan x+y$

Solution: Given function is $xy+y^2=\tan x+y$

Derivate function with respect to x, we have

$$\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}\tan x+\frac{d}{dx}y$$

$$x\frac{d}{dx}y+y\frac{d}{dx}x+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

[Solving first term using Product Rule]

$$x\frac{dy}{dx}+y+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$x\frac{dy}{dx}+2y\frac{dy}{dx}-\frac{dy}{dx}=\sec^2 x-y$$

$$(x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$

$$\frac{dy}{dx}=\frac{\sec^2 x-y}{x+2y-1}$$

5. $x^2+xy+y^2=100$

Solution: Given function is $x^2+xy+y^2=100$

Derivate function with respect to x, we have

$$\frac{d}{dx}x^2+\frac{d}{dx}xy+\frac{d}{dx}y^2=\frac{d}{dx}100$$

$$2x+\left(x\frac{d}{dx}y+y\frac{d}{dx}x\right)+2y\frac{dy}{dx}=0$$

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$(x+2y)\frac{dy}{dx}=-2x-y$$

6. $x^3+x^2y+xy^2+y^3=81$

Solution: Given function is $x^3+x^2y+xy^2+y^3=81$

Derivate function with respect to x, we have

$$\frac{d}{dx}x^3+\frac{d}{dx}x^2y+\frac{d}{dx}xy^2+\frac{d}{dx}y^3=\frac{d}{dx}81$$

$$3x^2+\left(x^2\frac{dy}{dx}+y.\frac{d}{dx}x^2\right)+x\frac{d}{dx}y^2+y^2\frac{d}{dx}x+3y^2\frac{dy}{dx}=0$$ (using product rule)

$$3x^2+x^2\frac{dy}{dx}+y.2x+x.2y\frac{dy}{dx}+y^2.1+3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(x^2+2xy+3y^2\right)=-3x^2-2xy-y^2$$

$$\frac{dy}{dx}=\frac{\left(3x^2+2xy+y^2\right)}{x^2+2xy+3y^2}$$

7. $\sin^2 y + \cos xy = \pi$

Solution: Given function is $\sin^2 y + \cos xy = \pi$

Derivate function with respect to x, we have

$$\frac{d}{dx}(\sin y)^2 + \frac{d}{dx}\cos xy = \frac{d}{dx}(\pi)$$

$$2\sin y \frac{d}{dx}\sin y - \sin xy \frac{d}{dx}(xy) = 0$$

$$2\sin y \cos y \frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y.1\right) = 0$$

$$\sin 2y \frac{dy}{dx} - x\sin xy \frac{dy}{dx} - y\sin xy = 0$$

$$(\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

8. $\sin^2 x + \cos^2 y = 1$

Solution: Given function is $\sin^2 x + \cos^2 x = 1$
Derivate function with respect to x, we have

$$\frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos x)^2 = \frac{d}{dx}(1)$$

$$2\sin x \frac{d}{dx}\sin x + 2\cos y \frac{d}{dx}\cos y = 0$$

$$2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$

$$\sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$-\sin 2y\frac{dy}{dx}=-\sin 2x$

$\frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$

9. $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution: Given function is

$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Step 1: Simplify the given function,

Put $x=\tan\theta$, we have

$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$=\sin^{-1}(\sin 2\theta)=2\theta$

Result in terms of x, we get

$y=2\tan^{-1}x$

Step 2: Derivative the function

$\frac{dy}{dx}=2.\frac{1}{1+x^2}=\frac{2}{1+x^2}$

10. $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Solution: Given function is

$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Step 1: Simplify the given function,

$$y=\tan^{-1}\left(\frac{3\tan\theta-\tan^{3}\theta}{1-3\tan^{2}\theta}\right)$$

$$y=\tan^{-1}(\tan 3\theta)=3\theta$$

Result in terms of x, we get

$$y=3\tan^{-1}x$$

Step 2: Derivative the function

$$\frac{dy}{dx}=3.\frac{1}{1+x^{2}}=\frac{3}{1+x^{2}}$$

11. $y=\cos^{-1}\left(\frac{1-x^{2}}{1+x^{2}}\right), 0<x<1$

Solution: Given function is

$$y=\cos^{-1}\left(\frac{1-x^{2}}{1+x^{2}}\right), 0<x<1$$

Step 1: Simplify the given function,

Put $x=\tan\theta$

$$y=\cos^{-1}\left(\frac{1-\tan^{2}\theta}{1+\tan^{2}\theta}\right)$$

$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$

Step 2: Derivative the function

$$\frac{dy}{dx}=2.\frac{1}{1+x^{2}}=\frac{2}{1+x^{2}}$$

12. $y=\sin^{-1}\left(\frac{1-x^{2}}{1+x^{2}}\right), 0<x<1$

Solution: Given function is $y=\cos^{-1}\left(\frac{1-x^{2}}{1+x^{2}}\right), 0<x<1$

Step 1: Simplify the given function,

Put $x=\tan\theta$

$$y=\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$=\sin^{-1}(\cos 2\theta)$

$=\sin^{-1}\sin\left(\frac{\pi}{2}-2\theta\right)=\frac{\pi}{2}-2\theta$

$=\frac{\pi}{2}-2\tan^{-1}x$

Step 2: Derivative the function

$\frac{dy}{dx}=0-2.\frac{1}{1+x^2}=\frac{-2}{1+x^2}$ (Derivative of a constant is always revert a value zero)

13. $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$

Solution: Given function is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$

Step 1: Simplify the given function,

Put $x=\tan\theta$

$$y=\cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$=\cos^{-1}(\cos 2\theta)$

$=\cos^{-1}\cos\left(\frac{\pi}{2}-2\theta\right)=\frac{\pi}{2}-2\theta$

$=\frac{\pi}{2}-2\tan^{-1}x$

Step 2: Derivative the function

$\frac{dy}{dx} = 0 - 2.\frac{1}{1+x^2} = \frac{-2}{1+x^2}$ (Derivative of a constant is zero)

14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Solution: Given function is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Step 1: Simplify the given function,

Put $x = \sin\theta$

$y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

= $\sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

= $\sin^{-1}\left(2\sin\theta\cos\theta\right)$

= $\sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

Step 2: Derivative the function

$\frac{dy}{dx} = 2.\frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$

15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Solution: Given function is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Step 1: Simplify the given function,

Put $x = \cos\theta$

$$y = y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$$

$= y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= \sec^{-1}(\sec 2\theta)$

$= 2\theta = 2\cos^{-1}x$

Step 2: Derivative the function

$$\frac{dy}{dx} = 2.\frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Exercise 5.4

Page No: 174

Differentiate the functions with respect to x in Exercise 1 to 10.

1. $\frac{e^x}{\sin x}$

Solution: Let $y = \frac{e^x}{\sin x}$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx} e^x - e^x \frac{d}{dx} \sin x}{\sin^2 x}$$

[Using quotient rule]

$$= \frac{\sin x e^x - e^x \cos x}{\sin^2 x}$$

$$= \frac{e^x (\sin x - \cos x)}{\sin^2 x}$$

2. $e^{\sin^{-1} x}$

Solution: Let $y = e^{\sin^{-1} x}$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = e^{\sin^{-1} x} . \frac{d}{dx} \sin^{-1} x$$

$$= e^{\sin^{-1} x} . \frac{1}{\sqrt{1-x^2}}$$

$$\left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

3. e^{x^3}

Solution: Let $y = e^{x^3} = e^{(x^3)}$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = e^{\left(x^3\right)} \frac{d}{dx} x^3$$

$$= e^{\left(x^3\right)}.3x^2 = 3x^2.e^{\left(x^3\right)}$$

$$\left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

4. $\sin\left(\tan^{-1} e^{-x}\right)$

Solution: Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = \cos\left(\tan^{-1} e^{-x}\right) \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$\left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1+\left(e^{-x}\right)^2} \frac{d}{dx} e^{-x}$$

$$\left[\because \frac{d}{dx} \tan^{-1} f(x) = \frac{1}{\left(f(x)\right)^2} \frac{d}{dx} f(x)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1+e^{-2x}} e^{-x} \frac{d}{dx}(-x)$$

$$= -\frac{e^{-x} \cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

5. $\log\left(\cos e^{x}\right)$

Solution: Let $y = \log\left(\cos e^{x}\right)$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{\cos e^x}\frac{d}{dx}(\cos e^x) \quad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}e^x \quad \left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -(\tan e^x)e^x = -e^x(\tan e^x)$$

6. $e^x + e^{x^2} + \ldots\ldots + e^{x^5}$

Solution: Let $y = e^x + e^{x^2} + \ldots.. + e^{x^5}$
Define the given function for 5 terms,
Let us say, y = $e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}e^x + \frac{d}{dx}e^{x^2} + \frac{d}{dx}e^{x^3} + \frac{d}{dx}e^{x^4} + \frac{d}{dx}e^{x^5}$$

$$= e^x + e^{x^2}\frac{d}{dx}x^2 + e^{x^3}\frac{d}{dx}x^3 + e^{x^4}\frac{d}{dx}x^4 + e^{x^5}\frac{d}{dx}x^5$$

$$= e^x + e^{x^2}.2x + e^{x^3}.3x^2 + e^{x^4}.4x^3 + e^{x^5}.5x^4$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3.e^{x^4} + 5x^4.e^{x^5}$$

7. $\sqrt{e^{\sqrt{x}}}, x > 0$

Solution: Let $y = \sqrt{e^{\sqrt{x}}}$

or y = $\left(e^{\sqrt{x}}\right)^{\frac{1}{2}}$

Differentiate the functions with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\sqrt{x}}\right)^{\frac{-1}{2}}\frac{d}{dx}e^{\sqrt{x}}$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$= \dfrac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \dfrac{d}{dx}\sqrt{x}$

$= \dfrac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \dfrac{1}{2\sqrt{x}}$

$= \dfrac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}}$

8. $\log(\log x), x > 1$

Solution: Let $y = \log(\log x)$

Differentiate the functions with respect to x, we get

$\dfrac{dy}{dx} = \dfrac{1}{\log x}\dfrac{d}{dx}(\log x)$

$= \dfrac{1}{\log x}.\dfrac{1}{x} = \dfrac{1}{x\log x}$

9. $\dfrac{\cos x}{\log x}, x > 0$

Solution: Let $y = \dfrac{\cos x}{\log x}$

Differentiate the functions with respect to x, we get

$\dfrac{dy}{dx} = \dfrac{\log x \dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(\log x)}{(\log x)^2}$ [By quotient rule]

$= \dfrac{\log x(-\sin x) - \cos x.\dfrac{1}{x}}{(\log x)^2}$

$= \dfrac{-\left(\sin x \log x + \dfrac{\cos x}{x}\right)}{(\log x)^2}$

$= \dfrac{-(x \sin x \log x + \cos x)}{x(\log x)^2}$

10. $\cos\left(\log x + e^x\right), x > 0$

Solution: Let $y = \cos\left(\log x + e^x\right)$

Differentiate the functions with respect to x, we get

$\dfrac{dy}{dx} = -\sin\left(\log x + e^x\right)\dfrac{d}{dx}\left(\log x + e^x\right)$

$= -\sin\left(\log x + e^x\right).\left(\dfrac{1}{x} + e^x\right)$

$= \left(\dfrac{1}{x} + e^x\right)\sin\left(\log x + e^x\right)$

Exercise 5.5 Page No: 178

Differentiate the functions with respect to x in Exercise 1 to 5.

1. $\cos x \cos 2x \cos 3x$

Solution: Let $y = \cos x \cos 2x \cos 3x$

Taking logs on both sides, we get

$$\log y = \log(\cos x \cos 2x \cos 3x)$$

$$= \log \cos x + \log \cos 2x + \log \cos 3x$$

Now,

$$\frac{d}{dx}\log y = \frac{d}{dx}\log \cos x + \frac{d}{dx}\log \cos 2x + \frac{d}{dx}\log \cos 3x$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}\frac{d}{dx}\cos x + \frac{1}{\cos 2x}\frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}\frac{d}{dx}\cos 3x$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-\sin 2x)\frac{d}{dx}2x + \frac{1}{\cos 3x}(-\sin 3x)\frac{d}{dx}3x$$

$$\frac{1}{y}.\frac{dy}{dx} = -\tan x - (\tan 2x)2 - \tan 3x(3)$$

$$\frac{dy}{dx} = -y(\tan x + 2\tan 2x + 3\tan 3x)$$

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$$ [using value of y]

2. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution: Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$= \left(\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right)^{\frac{1}{2}}$

Taking logs on both sides, we get

$$\log y = \frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1)+\frac{1}{x-2}\frac{d}{dx}(x-2)-\frac{1}{x-3}\frac{d}{dx}(x-3)-\frac{1}{x-4}\frac{d}{dx}(x-4)-\frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$ [using the value of y]

3. $(\log x)^{\cos x}$

Solution: Let $y = (\log x)^{\cos x}$

Taking logs on both sides, we get

$$\log y = \log(\log x)^{\cos x} = \cos x \log(\log x)$$

$$\frac{d}{dx}\log y = \frac{d}{dx}\left[\cos x \log(\log x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\log(\log x)+\log(\log x)\frac{d}{dx}\cos x$$ [By Product rule]

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{1}{\log x}\frac{d}{dx}(\log x)+\log(\log x)(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\log x}.\frac{1}{\log x}-\sin x\log(\log x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{\log x}-\sin x\log(\log x)\right]$$

$= (\log x)^{\cos x}\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$

4. $x^x - 2^{\sin x}$

Solution: Let $y = x^x - 2^{\sin x}$

Put $u = x^x$ and $v = 2^{\sin x}$

$y = u - v$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ....(1)

Now, $u = x^x$

$\log u = \log x^x = x \log x$

$\frac{d}{dx}\log u = \frac{d}{dx}(x \log x)$

$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log x + \log x \frac{d}{dx}x$

$\frac{1}{u}\frac{du}{dx} = 1 + \log x$

$\frac{du}{dx} = u(1 + \log x)$

$\frac{du}{dx} = x^x(1 + \log x)$ ....(2)

Again, $v = 2^{\sin x}$

$\frac{dv}{dx} = \frac{d}{dx}2^{\sin x}$

$\frac{dv}{dx} = 2^{\sin x}\log 2 \frac{d}{dx}\sin x \quad \left[\because \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a \frac{d}{dx}f(x)\right]$

$$\frac{dv}{dx} = 2^{\sin x}(\log 2).\cos x = \cos x.2^{\sin x}\log 2$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = x^x(1+\log x) - \cos x.2^{\sin x}\log 2$$

5. $(x+3)^2(x+4)^3(x+5)^4$

Solution: Let $y = (x+3)^2(x+4)^3(x+5)^4$

Taking logs on both sides, we get

$$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)^4$$

Now,

$$\frac{d}{dx}\log y = 2\frac{d}{dx}\log(x+3) + 3\frac{d}{dx}\log(x+4) + 4\frac{d}{dx}\log(x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x+3}\frac{d}{dx}(x+3) + 3\frac{1}{x+4}\frac{d}{dx}(x+4) + 4\frac{1}{x+5}\frac{d}{dx}(x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

$$\frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

(using value of y)

Differentiate the functions with respect to x in Exercise 6 to 11.

6. $\left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Solution: Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Put $\left(x+\frac{1}{x}\right)^x = u$ and $x^{\left(x+\frac{1}{x}\right)} = v$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ....(1)

Now $u = \left(x+\frac{1}{x}\right)^x$

$$\log u = \log\left(x+\frac{1}{x}\right)^x = x\log\left(x+\frac{1}{x}\right)$$

$$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x+\frac{1}{x}\right) + \log\left(x+\frac{1}{x}\right).1$$

$$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{\left(x+\frac{1}{x}\right)}\left(x-\frac{1}{x^2}\right) + \log\left(x+\frac{1}{x}\right).1$$

$$\frac{du}{dx} = u\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right]$$

= $\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right]$ ....(2)

Again $v = x^{\left(x+\frac{1}{x}\right)}$

$$\log v = \log x^{\left(x+\frac{1}{x}\right)} = \left(x+\frac{1}{x}\right)\log x$$

$$\frac{1}{v}\frac{dv}{dx} = \left(x+\frac{1}{x}\right).\frac{1}{x} + \log x.\left(\frac{-1}{x^2}\right)$$

$$\frac{dv}{dx}=v\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

$$\frac{dv}{dx}=x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$ ....(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

7. $(\log x)^x+x^{\log x}$

Solution: Let $y=(\log x)^x+x^{\log x}$ = $u+v$ where $u=(\log x)^x$ and $v=x^{\log x}$

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$ .... (1)

Now $u=(\log x)^x$

$$\log u=\log(\log x)^x=x\log(\log x)$$

$$\frac{d}{dx}\log u=\frac{d}{dx}[x\log(\log x)]$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}[\log(\log x)]+\log(\log x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{\log x}\frac{d}{dx}\log x+\log(\log x).1$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{\log x}\frac{1}{x}+\log(\log x)$$

$$\frac{du}{dx}=u\left[\frac{1}{\log x}+\log(\log x)\right]$$

$$\frac{du}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$$ .... (2)

Again $v = x^{\log x}$

$$\log v = \log x^{\log x} = \log x.\log x = (\log x)^2$$

$$\frac{d}{dx}\log v = \frac{d}{dx}(\log x)^2$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\frac{d}{dx}(\log x)$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x.\frac{1}{x}$$

$$\frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) = x^{\log x}.\frac{2}{x}\log x$$

$$\frac{dv}{dx} = 2x^{\log x - 1}\log x \quad ....(3)$$

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + 2x^{\log x-1}\log x$$

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1+\log x\log(\log x)}{\log x}\right] + 2x^{\log x-1}\log x$$

$$\frac{dy}{dx} = (\log x)^{x-1}(1+\log x\log(\log x)) + 2x^{\log x-1}\log x$$

8. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution: Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$ = $u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad .... (1)$$

Now $u = (\sin x)^x$

$$\log u = \log(\sin x)^x = x\log(\sin x)$$

$$\frac{d}{dx}\log u = \frac{d}{dx}\left[x\log(\sin x)\right]$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(\sin x)\right] + \log(\sin x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\frac{d}{dx}\sin x + \log(\sin x).1$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\cos x + \log(\sin) = x\cot x + \log\sin x$$

$$\frac{du}{dx} = u\left[x\cot x + \log\sin x\right]$$

$$\frac{du}{dx} = (\sin x)^x\left[x\cot x + \log\sin x\right]$$ .... (2)

Again $v = \sin^{-1}\sqrt{x}$

$$\log v = \log\sin^{-1}\sqrt{x}$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\sqrt{x} \quad \left[\because \frac{d}{dx}\sin^{-1} f(x) = \frac{1}{\sqrt{1-\left(f(x)\right)^2}}\frac{d}{dx}f(x)\right]$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x-x^2}}$$ ....(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = (\sin x)^x\left[x\cot x + \log\sin x\right] + \frac{1}{2\sqrt{x-x^2}}$$

9. $x^{\sin x}+(\sin x)^{\cos x}$

Solution: Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Put $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$, we get $y=u+v$

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ .... (1)

Now $u=x^{\sin x}$

$\log u=\log x^{\sin x}$ = $\sin x\log x$

$\frac{d}{dx}\log u=\frac{d}{dx}(\sin x\log x)$

$\frac{1}{u}\frac{du}{dx}=\sin x\frac{d}{dx}\log x+\log x\frac{d}{dx}\sin x$

$\frac{1}{u}\frac{du}{dx}=\sin x\frac{1}{x}+\log x(\cos x)$

$\frac{du}{dx}=u\left(\frac{\sin x}{x}+\cos x\log x\right)$

$\frac{du}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\log x\right)$ ..... (2)

Again $v=(\sin x)^{\cos x}$

$\log v=\log(\sin x)^{\cos x}$ = $\cos x\log\sin x$

$\frac{d}{dx}\log v=\frac{d}{dx}[\cos x\log(\sin x)]$

$\frac{1}{v}\frac{dv}{dx}=\cos x\frac{d}{dx}\log\sin x+\log\sin x\frac{d}{dx}\cos x$

$$\frac{1}{v}\frac{dv}{dx}=\cos x\frac{1}{\sin x}\frac{d}{dx}\sin x+\log\sin x(-\sin x)$$

$$\frac{1}{v}\frac{dv}{dx}=\cot x.\cos x-\sin x\log\sin x$$

$$\frac{dv}{dx}=v(\cot x\cos x-\sin x\log\sin x)$$

$$\frac{dv}{dx}=(\sin x)^{\cos x}(\cot x.\cos x-\sin x\log\sin x)$$ (using value of v) ……….(3)

Put values from (2) and (3) in (1),

$$\frac{dy}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\log x\right)+(\sin x)^{\cos x}(\cot x.\cos x-\sin x\log\sin x)$$

10. $x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Solution: Let $x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Put $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$, we get $y=u+v$

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$ ……….(1)

Now $u=x^{x\cos x}$

$\log u=\log x^{x\cos x}$ = $x\cos x\log x$

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\cos x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x).\cos x\log x+x\frac{d}{dx}(\cos x)\log x+x\cos x\frac{d}{dx}(\log x)$$

$$\frac{1}{u}\frac{du}{dx}=1.\cos x\log x+x(-\sin x)\log x+x\cos x\frac{1}{x}$$

$$\frac{du}{dx} = u\left(\cos x \log x - x\sin x \log x + \cos x\right)$$

$$\frac{du}{dx} = x^{x\cos x}\left(\cos x \log x - x\sin x \log x + \cos x\right)$$ .... (2)

Again $v = \frac{x^2+1}{x^2-1}$

$$\frac{dv}{dx} = \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{\left(x^2-1\right)2x - \left(x^2+1\right)2x}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2}$$ ....(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = x^{x\cos x}\left(\cos x \log x - x\sin x \log x + \cos x\right) + \frac{-4x}{\left(x^2-1\right)^2}$$

11. $\left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Solution: Let $y = \left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Put $u = \left(x\cos x\right)^x$ and $v = \left(x\sin x\right)^{\frac{1}{x}}$, we get $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$ .... (1)

Now $u = \left(x\cos x\right)^x$

$\log u = \log (x \cos x)^x = x \log (x \cos x)$

$\log u = x(\log x + \log \cos x)$

$\frac{d}{dx}\log u = \frac{d}{dx}\{x(\log x + \log \cos x)\}$

$\frac{1}{u}\frac{du}{dx} = x\left[\frac{1}{x} + \frac{1}{\cos x}.(-\sin x)\right] + (\log x + \log \cos x).1$

$\frac{du}{dx} = u[1 - x \tan x + \log (x \cos x)]$

$\frac{du}{dx} = (x \cos x)^x [1 - x \tan x + \log (x \cos x)]$ .... (2)

Again $v = (x \sin x)^{\frac{1}{x}}$

$\log v = \log (x \sin x)^{\frac{1}{x}} = \frac{1}{x}\log (x \sin x)$

$\log v = \frac{1}{x}(\log x + \log \sin x)$

$\frac{d}{dx}\log v = \frac{d}{dx}\left\{\frac{1}{x}(\log x + \log \sin x)\right\}$

$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x}.\cos x\right] + (\log x + \log \sin x)\left(\frac{-1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log (x \sin x)}{x^2}\right]$

$\frac{dv}{dx} = (x \sin x)^{\frac{1}{x}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log (x \sin x)}{x^2}\right]$ ....(3)

Put the values from (2) and (3) in (1)

$$\frac{dy}{dx}=(x\cos x)^{x}\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$$

Find $\frac{dy}{dx}$ in the following Exercise 12 to 15

12. $x^{y}+y^{x}=1$

Solution: Given: $x^{y}+y^{x}=1$

$u+v=1,$ where $u=x^{y}$ and $v=y^{x}$

$$\frac{d}{dx}u+\frac{d}{dx}v=\frac{d}{dx}1$$

$$\frac{du}{dx}+\frac{dv}{dx}=0 \quad \text{.......... (1)}$$

Now $u=x^{y}$

$$\log u=\log x^{y}=y\log x$$

$$\frac{d}{dx}\log u=\frac{d}{dx}(y\log x)$$

$$\frac{1}{u}\frac{du}{dx}=y\frac{d}{dx}\log x+\log x\frac{dy}{dx}$$

$$\frac{1}{u}\frac{du}{dx}=y.\frac{1}{x}+\log x.\frac{dy}{dx}$$

$$\frac{du}{dx}=u\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)$$

$$\frac{du}{dx}=x^{y}\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)=x^{y}\frac{y}{x}+x^{y}\log x.\frac{dy}{dx}$$

$$\frac{du}{dx}=x^{y-1}y+x^{y}\log x.\frac{dy}{dx} \quad \text{.......... (2)}$$

Again $v=y^{x}$

$$\log v = \log y^x = x \log y$$

$$\frac{d}{dx}\log v = \frac{d}{dx}(x \log y)$$

$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}\log y + \log y \frac{d}{dx}x$$

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1$$

$$\frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

$$\frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x \frac{x}{y}\frac{dy}{dx} + y^x \log y$$

$$\frac{dv}{dx} = y^{x-1}x\frac{dy}{dx} + y^x \log y \quad \text{..........(3)}$$

Put values from (2) and (3) in (1),

$$x^{y-1}y + x^y \log x.\frac{dy}{dx} + y^{x-1}x\frac{dy}{dx} + y^x \log y = 0$$

$$\frac{dy}{dx}\left(x^y \log x + y^{x-1}x\right) = -x^{y-1}y - y^x \log y$$

$$\frac{dy}{dx} = \frac{-\left(x^{y-1}y - y^x \log y\right)}{x^y \log x + y^{x-1}x}$$

13. $y^x = x^y$

Solution: Given: $y^x = x^y$

$$x^y = y^x$$

$$\log x^y = \log y^x$$

$$y \log x = x \log y$$

$$\frac{d}{dx}(y\log x)=\frac{d}{dx}(x\log y)$$

$$y.\frac{1}{x}+\log x.\frac{dy}{dx}=x.\frac{1}{y}.\frac{dy}{dx}+\log y.1$$

$$\left(\log x-\frac{x}{y}\right)\frac{dy}{dx}=\log y-\frac{y}{x}$$

$$\left(\frac{y\log x-x}{y}\right)\frac{dy}{dx}=\frac{x\log y-y}{x}$$

$$\frac{dy}{dx}=\frac{y(x\log y-y)}{x(y\log x-x)}$$

14. $(\cos x)^y=(\cos y)^x$

Solution: Given: $(\cos x)^y=(\cos y)^x$

$$\log(\cos x)^y=\log(\cos y)^x$$

$$y\log\cos x=x\log\cos y$$

$$\frac{d}{dx}(y\log\cos x)=\frac{d}{dx}(x\log\cos y)$$

$$y\frac{d}{dx}\log\cos x+\log\cos x\frac{dy}{dx}=x\frac{d}{dx}\log\cos y+\log\cos y\frac{d}{dx}x$$

$$y\frac{1}{\cos x}\frac{d}{dx}\cos x+\log\cos x\frac{dy}{dx}=x\frac{1}{\cos y}\frac{d}{dx}\cos y+\log\cos y$$

$$y\frac{1}{\cos x}(-\sin x)+\log\cos x\frac{dy}{dx}=x\frac{1}{\cos y}\left(-\sin y\frac{dy}{dx}\right)+\log\cos y$$

$$-y\tan x+\log\cos x.\frac{dy}{dx}=-x\tan y.\frac{dy}{dx}+\log\cos y$$

$$x\tan y\frac{dy}{dx}+\log\cos x.\frac{dy}{dx}=y\tan x+\log\cos y$$

$$\frac{dy}{dx}(x\tan y+\log\cos x)=y\tan x+\log\cos y$$

$$\frac{dy}{dx}=\frac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$$

15. $xy=e^{x-y}$

Solution: Given: $xy=e^{x-y}$

$$\log xy=\log e^{x-y}$$

$$\log x+\log y=(x-y)\log e$$

$$\log x+\log y=(x-y)\quad [\because \log e=1]$$

$$\frac{d}{dx}\log x+\frac{d}{dx}\log y=\frac{d}{dx}(x-y)$$

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}+\frac{dy}{dx}=1-\frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y}+1\right)=\frac{x-1}{x}$$

$$\frac{dy}{dx}\left(\frac{1+y}{y}\right)=\frac{x-1}{x}$$

$$\frac{dy}{dx}=\frac{y(x-1)}{x(1+y)}$$

16. Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ **and hence** $f'(1)$.

Solution: Given: $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ .... (1)

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

$$\frac{1}{f(x)}\frac{d}{dx}f(x)=\frac{1}{1+x}\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\frac{d}{dx}(1+x^8)$$

$$\frac{1}{f(x)}f'(x)=\frac{1}{1+x}.1+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}.4x^3+\frac{1}{1+x^8}8x^7$$

$$f'(x)=f(x)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Put the value of f(x) from (1),

$$f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Now, Find for f'(1):

$$f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$$

$$f'(1)=(2)(2)(2)(2)\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$f'(1)=16\left[\frac{15}{2}\right]$$

= 8 x 15

= 120

17. Differentiate $(x^2-5x+8)(x^3+7x+9)$ **in three ways mentioned below:**
(i) by using product rule.
(ii) by expanding the product to obtain a single polynomial
(iii) by logarithmic differentiation.
Do they all give the same answer?

Solution: Let $y=(x^2-5x+8)(x^3+7x+9)$
(i) using product rule:

$$\frac{dy}{dx}=\left(x^2-5x+8\right)\frac{d}{dx}\left(x^3+7x+9\right)+\left(x^3+7x+9\right)\frac{d}{dx}\left(x^2-5x+8\right)$$
$$\frac{dy}{dx}=\left(x^2-5x+8\right)\left(3x^2+7\right)+\left(x^3+7x+9\right)\left(2x-5\right)$$

$$\frac{dy}{dx}=3x^4+7x^2-15x^3-35x+24x^2+56+2x^4-5x^3+14x^2-35x+18x-45$$

$$\frac{dy}{dx}=5x^4-20x^3+45x^2+11$$

(ii) Expand the product to obtain a single polynomial

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$
$$y=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$y=x^5-5x^4+15x^3-26x^2+11x+72$$

$$\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(iii) Logarithmic differentiation

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$
$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

$$\frac{d}{dx}\log y=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\left(2x-5\right)+\frac{1}{x^3+7x+9}\left(3x^2+7\right)$$

$$\frac{dy}{dx}=y\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\frac{dy}{dx}=y\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\frac{dy}{dx}=y\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=y\left[\frac{2x^4+14x^2+18x-5x^3-35x-45+3x^4-15x^3+24x^2+7x^2-35x+56}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=y\left[\frac{5x^4-20x^3+45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{5x^4-20x^3+45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$$ [using value of y]

$$\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

Therefore, the value of dy/dx is same obtained by three different methods.

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}$$

in two ways–first by repeated application of product rule, second by logarithmic differentiation.

Solution: Given u, v and w are functions of x.

To Prove: $\frac{d}{dx}(u.v.w)=\frac{du}{dx}.v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$

Way 1: By repeated application of product rule

L.H.S.

$$\frac{d}{dx}(u.v.w)=\frac{d}{dx}[(uv).w]$$

$$=uv\frac{d}{dx}w+w\frac{d}{dx}(uv)$$

$$=uv\frac{dw}{dx}+w\left[u\frac{d}{dx}v+v\frac{d}{dx}u\right]$$

$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$= \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

= R.H.S

Hence proved.

Way 2: By Logarithmic differentiation

Let $y = uvw$

$\log y = \log(u.v.w)$

$\log y = \log u + \log v + \log w$

$\frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$

$\frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

Put y=uvw, we get

$\frac{d}{dx}(u.v.w) = uvw\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

$\frac{d}{dx}(u.v.w) = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

Hence proved.

Exercise 5.6

If x and y are connected parametrically by the equations given in Exercise 1 to 10, without eliminating the parameter, find dy/dx.

1. $x = 2at^2, y = at^4$

Solution: Given functions are $x = 2at^2$ and $y = at^4$

$$\frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right)$$

$$\frac{dx}{dt} = 2a\frac{d}{dt}\left(t^2\right)$$

= $2a.2t = 4at$ and

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right)$$

$$\frac{dy}{dt} = a\frac{d}{dt}\left(t^4\right) = a.4t^3 = 4at^3$$

Now,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$$

2. $x = a\cos\theta, y = b\cos\theta$

Solution: Given functions are $x = a\cos\theta$ and $y = b\cos\theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right)$$

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\left(\cos\theta\right)$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right)$$

$$\frac{dy}{d\theta}=b\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dy}{d\theta}=-b\sin\theta$$

Now,

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-a\sin\theta}{-b\sin\theta}=\frac{b}{a}$$

3. $x=\sin t, y=\cos 2t$

Solution: Given functions are $x=\sin t$ and $y=\cos 2t$

$\frac{dx}{dt}=\cos t$ and

$$\frac{dy}{dt}=-\sin 2t\frac{d}{dt}(2t) = -2\sin 2t$$

Now,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-2\sin 2t}{\cos t}=\frac{-2\times 2\sin t\cos t}{\cos t}=-4\sin t$$

4. $x=4t, y=\frac{4}{t}$

Solution: Given functions are $x=4t$ and $y=\frac{4}{t}$

$$\frac{dx}{dt}=\frac{d}{dt}(4t) = 4\frac{d}{dt}t=4$$

and

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)$$

$$=\frac{t\frac{d}{dt}4-4\frac{d}{dt}t}{t^2}$$

$$\Rightarrow \frac{dy}{dt}=\frac{t\times 0-4\times 1}{t^2}=-\frac{4}{t^2}$$

Now,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\frac{4}{t^2}}{4}=\frac{-1}{t^2}$$

5. $x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Solution: Given functions are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\cos\theta-\frac{d}{d\theta}\cos 2\theta$$

$$\frac{dx}{d\theta}=-\sin\theta-(-\sin 2\theta)\frac{d}{d\theta}2\theta$$

$$\frac{dx}{d\theta}=-\sin\theta+(-\sin 2\theta)2$$

$$\frac{dx}{d\theta}=2\sin 2\theta-\sin\theta$$

And

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\sin\theta-\frac{d}{d\theta}\sin 2\theta$$

$$\frac{dy}{d\theta}=\cos\theta-\cos 2\theta\frac{d}{d\theta}2\theta$$

$$\frac{dy}{d\theta}=\cos\theta-\cos 2\theta\times 2$$

$$\frac{dy}{d\theta}=\cos\theta-2\cos 2\theta$$

Now $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$

6. $x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Solution: Given functions are $x=a(\theta-\sin\theta)$ and $y=a(1+\cos\theta)$

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\theta - \frac{d}{d\theta}\sin\theta\right]$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}\cos\theta\right]$$

$$\frac{dy}{d\theta} = a(0 - \sin\theta)$$

$$= -a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}$$

$$= -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= -\cot\frac{\theta}{2}$$

7. $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}, y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Solution: Given functions are $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\frac{dx}{dt}=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t.\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\left(\sqrt{\cos 2t}\right)^2}$$ [By quotient rule]

$$=\frac{\sqrt{\cos 2t}.3\sin^2 t\frac{d}{dt}(\sin t)-\sin^3 t.\frac{1}{2}(\cos 2t)^{\frac{-1}{2}}\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\sin^2 t\cos t\cos 2t+\sin^3 t.\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{\sin^2 t\cos t\left(3\cos 2t+2\sin^2 t\right)}{(\cos 2t)^{\frac{3}{2}}}$$

And $$\frac{dy}{dt}=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t.\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\left(\sqrt{\cos 2t}\right)^2}$$ [By quotient rule]

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2}(\cos 2t)^{\frac{-1}{2}}\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t(-\sin t)-\frac{\cos^3 t}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos^2 t\sin t\cos 2t+\cos^3 t.\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$= \dfrac{-3\cos^2 t \sin t \cos 2t + \cos^3 t . 2\sin t \cos t}{(\cos 2t)^{\frac{3}{2}}}$

$= \dfrac{\sin t \cos^2 t\left(2\cos^2 t - 3\cos 2t\right)}{(\cos 2t)^{\frac{3}{2}}}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dfrac{\sin t \cos^2 t\left(2\cos^2 t - 3\cos 2t\right)}{(\cos 2t)^{\frac{3}{2}}}}{\dfrac{\sin^2 t \cos t\left(3\cos 2t + 2\sin^2 t\right)}{(\cos 2t)^{\frac{3}{2}}}}$

$= \dfrac{\cos t\left[2\cos^2 t - 3\left(2\cos^2 t - 1\right)\right]}{\sin t\left[3\left(1 - 2\sin^2 t\right) + 2\sin^2 t\right]}$

$= \dfrac{\cos t\left(3 - 4\cos^2 t\right)}{\sin t\left(3 - 4\sin^2 t\right)}$

$= \dfrac{-\left(4\cos^2 t - 3\cos t\right)}{3\sin t - 4\sin^3 t}$

$= \dfrac{-\cos 3t}{\sin 3t} = -\cot 3t$

8. $x = a\left(\cos t + \log \tan \dfrac{t}{2}\right), y = a\sin t$

Solution: Given functions are $x = a\left(\cos t + \log \tan \dfrac{t}{2}\right)$ and $y = a\sin t$

$\dfrac{dx}{dt} = a\left[-\sin t + \dfrac{1}{\tan \frac{t}{2}} \cdot \dfrac{d}{dt}\left(\tan \dfrac{t}{2}\right)\right]$

$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right]$

$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2}\right]$

$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$

$= a\left[-\sin t + \frac{1}{\sin t}\right]$

$= a\left[\frac{1}{\sin t} - \sin t\right] = a\left(\frac{1-\sin^2 t}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}$

and $\frac{dy}{dt} = a\cos t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)}$

$= \frac{\sin t}{\cos t} = \tan t$

9. $x = a\sec\theta, y = b\tan\theta$

Solution: Given functions are $x = a\sec\theta$ and $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ and

$\frac{dy}{d\theta} = \sec^2\theta$

Now,

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$=\frac{b\sec\theta}{a\tan\theta}$

$=\frac{b.\frac{1}{\cos\theta}}{a.\frac{\sin\theta}{\cos\theta}}$

$=\frac{b}{\cos\theta}.\frac{\cos\theta}{\sin\theta}$

$=\frac{b}{a\sin\theta}$

$=\frac{b}{a}\cos ec\theta$

10. $x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$

Solution: Given functions are $x=a(\cos\theta+\theta\sin\theta)$ and $y=a(\sin\theta-\theta\cos\theta)$

$$\frac{dx}{d\theta}=a(-\sin\theta+\theta\cos\theta+\sin\theta.1)$$

$=a\theta\cos\theta$

and

$$\frac{dy}{d\theta}=a[\cos\theta-\{\theta(-\sin\theta)+\cos\theta.1\}]$$

$=a[\cos\theta+\theta\sin\theta-\cos\theta]$

$=a\theta\sin\theta$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

11.

If $x=\sqrt{a^{\sin^{-1}t}}, y=\sqrt{a^{\cos^{-1}t}}$, **show that** $\frac{dy}{dx}=\frac{-y}{x}$.

Solution:

$x=\sqrt{a^{\sin^{-1}t}}=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$

$= a^{\frac{1}{2}\sin^{-1}t}$

and

$y=\sqrt{a^{\cos^{-1}t}}=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$= a^{\frac{1}{2}\cos^{-1}t}$

Now,

$$\frac{dx}{dt}=a^{\frac{1}{2}\sin^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}\sin^{-1}t\right)$$

$$= a^{\frac{1}{2}\sin^{-1}t}\log a\frac{1}{2}\frac{1}{\sqrt{1-t^2}}$$

And $\frac{dy}{dt}=a^{\frac{1}{2}\cos^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}\cos^{-1}t\right)$

$$= a^{\frac{1}{2}\cos^{-1}t}\log a\frac{1-}{2}\frac{1}{\sqrt{1-t^2}}$$

Now,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a^{\frac{1}{2}\cos^{-1}t}\log a\frac{1}{2}\frac{-1}{\sqrt{1-t^2}}}{a^{\frac{1}{2}\sin^{-1}t}\log a\frac{1}{2}\frac{1}{\sqrt{1-t^2}}}$$

$$= \frac{-a^{\frac{1}{2}\cos^{-1}t}}{a^{\frac{1}{2}\sin^{-1}t}} = \frac{-y}{x}$$

Exercise 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. x^2+3x+2

Solution: Let $y=x^2+3x+2$

First derivative:

$$\frac{dy}{dx}=2x+3\times1+0=2x+3$$

Second derivative:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=2\times1+0=2$$

2. x^{20}

Solution: Let $y=x^{20}$

Derivate y with respect to x, we get

$$\frac{dy}{dx}=20x^{19}$$

Derivate dy/dx with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=20\times19x^{18}=380x^{18}$$

3. $x\cos x$

Solution: Let $y=x\cos x$

$$\frac{dy}{dx}=x\frac{d}{dx}\cos x+\cos x\frac{d}{dx}x$$

= $-x\sin x+\cos x$

Now,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{d}{dx}(x\sin x)+\frac{d}{dx}\cos x$$

$= -\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right] - \sin x$

$= -(x\cos x + \sin x) - \sin x$

$= -x\cos x - \sin x - \sin x$

$= -x\cos x - 2\sin x$

$= -(x\cos x + 2\sin x)$

4. $\log x$

Solution: Let $y = \log x$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$

$\frac{d^2y}{dx^2} = (-1)x^{-2}$ $= \frac{-1}{x^2}$

5. $x^3\log x$

Solution: Let $y = x^3\log x$

$\frac{dy}{dx} = x^3\frac{d}{dx}\log x + \log x\frac{d}{dx}x^3$

$= x^3.\frac{1}{x} + \log x(3x^2)$

$= x^2 + 3x^2\log x$

Now,

$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 + 3x^2\log x)$

$= \frac{d}{dx}x^2 + 3\frac{d}{dx}(x^2 \log x)$

$= 2x + 3\left[x^2 \frac{d}{dx}\log x + \log x \frac{d}{dx}x^2\right]$

$= 2x + 3\left(x^2 . \frac{1}{x} + (\log x)2x\right)$

$= 2x + 3(x + 2x\log x)$

$= 2x + 3x + 6x\log x$

$= 5x + 6x\log x$

$= x(5 + 6\log x)$

6. $e^x \sin 5x$

Solution: Let $y = e^x \sin 5x$

$\frac{dy}{dx} = e^x \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}e^x$

$= e^x \cos 5x \frac{d}{dx}5x + \sin 5x e^x$

$= e^x \cos 5x \times 5 + e^x \sin 5x$

$= e^x (5\cos 5x + \sin 5x)$

Now,

$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(5\cos 5x + \sin 5x) + (5\cos 5x + \sin 5x)\frac{d}{dx}e^x$

$= e^x \{5(-\sin x) \times 5 + (\cos 5x) \times 5\} + (5\cos 5x + \sin 5x)e^x$

$= e^x (-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x)$

= $e^{x}\left(10\cos 5x-24\sin 5x\right)$

= $2e^{x}\left(5\cos 5x-12\sin 5x\right)$

7. $e^{6x}\cos 3x$

Solution: Let $y=e^{6x}\cos 3x$

$$\frac{dy}{dx}=e^{6x}\frac{d}{dx}\cos 3x+\cos 3x\frac{d}{dx}e^{6x}$$

= $e^{6x}\left(-\sin 3x\right)\frac{d}{dx}\left(3x\right)+\cos 3x.e^{6x}\frac{d}{dx}\left(6x\right)$

= $-e^{6x}\sin 3x\times 3+\cos 3x.e^{6x}\times 6$

= $e^{6x}\left(-3\sin 3x+6\cos 6x\right)$

Now,

$$\frac{d^2y}{dx^2}=e^{6x}\frac{d}{dx}\left(-3\sin 3x+6\cos 3x\right)+\left(-3\sin 3x+6\cos 3x\right)\frac{d}{dx}e^{6x}$$

= $e^{6x}\left(-3\cos 3x\times 3-6\sin 3x\times 3\right)+\left(-3\sin 3x+6\cos 3x\right)e^{6x}\times 6$

= $e^{6x}\left(-9\cos 3x-18\sin 3x-18\sin 3x+36\cos 3x\right)$

= $9e^{6x}\left(3\cos 3x-4\sin 3x\right)$

8. $\tan^{-1}x$

Solution: Let $y=\tan^{-1}x$

$$\frac{dy}{dx}=\frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$= \frac{(1+x^2)\frac{d}{dx}(1) - 1\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$

$= \frac{(1+x^2)\times 0 - 2x}{(1+x^2)^2}$

$= \frac{-2x}{(1+x^2)^2}$

9. $\log(\log x)$

Solution: Let $y = \log(\log x)$

$\frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}\log x$

$\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$

$= \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x\log x)\frac{d}{dx}(1) - 1\frac{d}{dx}(x\log x)}{(x\log x)^2}$

$= \frac{(x\log x)(0) - \left[x\frac{d}{dx}\log x + \log x\frac{d}{dx}x\right]}{(x\log x)^2}$

$= \frac{\left[x\frac{1}{x} + \log x \times 1\right]}{(x\log x)^2}$

$= \frac{[1+\log x]}{(x\log x)^2}$

10. $\sin(\log x)$

Solution: Let $y=\sin(\log x)$

$\frac{dy}{dx}=\cos(\log x)\frac{d}{dx}(\log x)$

$=\cos(\log x).\frac{1}{x}$

$=\frac{\cos(\log x)}{x}$

Now,

$\frac{d^2y}{dx^2}=\frac{x\frac{d}{dx}\cos(\log x)-\cos(\log x)\frac{d}{dx}x}{x^2}$

$=\frac{x[-\sin(\log x)]\frac{d}{dx}(\log x)-\cos(\log x)\times 1}{x^2}$

$=\frac{-x\sin(\log x)\frac{1}{x}-\cos(\log x)}{x^2}$

$=\frac{-[\sin(\log x)+\cos(\log x)]}{x^2}$

11. If $y=5\cos x-3\sin x$, **prove that** $\frac{d^2y}{dx^2}+y=0.$

Solution: Let $y=5\cos x-3\sin x$ …….(1)

$\frac{dy}{dx}=-5\sin x-3\cos x$

Now,

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$$

= $-(5\cos x - 3\sin x)$ = $-y$ [From (1)]

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

12. If $y = \cos^{-1} x$. Find $\frac{d^2y}{dx^2}$ in terms of y alone.

Solution: Given: $y = \cos^{-1} x$
or x = cos y ……….(1)

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

= $\frac{-1}{\sqrt{1-\cos^2 y}}$ [From (1)]

= $\frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y$ ……….(2)

Now,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y)$$

= $-\left[-\operatorname{cosec} y \cot y \frac{dy}{dx}\right]$

= $\operatorname{cosec} y \cot y(-\operatorname{cosec} y)$

= $-\operatorname{cosec}^2 y \cot y$ [From (2)]

13. If $y = 3\cos(\log x) + 4\sin(\log x)$, **show that** $x^2 y_2 + xy_1 + y = 0$.

Solution: Given function is

$y = 3\cos(\log x) + 4\sin(\log x)$ ....(1)

Derivate with respect to x, we get

$$\frac{dy}{dx} = y_1 = -3\sin(\log x)\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}\log x$$

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

$$= \frac{1}{x}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$

$$xy_1 = -3\sin(\log x) + 4\cos(\log x)$$

Now, derivate above equation once again

$$\frac{d}{dx}(xy_1) = -3\cos(\log x)\frac{d}{dx}\log x - 4\sin(\log x)\frac{d}{dx}\log x$$

$$x\frac{d}{dx}(y_1) + y_1\frac{d}{dx}x = -3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}$$

$$xy_2 + y_1 = -\frac{\left[3\cos(\log x) + 4\sin(\log x)\right]}{x}$$

$$x(xy_2 + y_1) = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$

$x(xy_2 + y_1) = -y$ [using equation (1)]

This implies, $x^2 y_2 + xy_1 + y = 0$

Hence proved.

14. If $y = Ae^{mx} + Be^{nx}$, show that
$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$

Solution:

To Prove: $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

$y = Ae^{mx} + Be^{nx}$ ….(1)

$\frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx)$ $\left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$

$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$ ….(2)

Find the derivate of equation (2)

$\frac{d^2y}{dx^2} = Ame^{mx}.m + Bne^{nx}.n$

$= Am^2e^{mx} + Bn^2e^{nx}$ ….(3)

Now, L.H.S.= $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

(Using equations (1), (2) and (3))

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)Ame^{mx} + Bne^{nx} + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} + Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$

= 0

= R.H.S.

Hence proved.

15. If $y=500e^{7x}$, **show that** $\frac{d^2y}{dx^2}=49y$.

Solution:

$y=500e^{7x}+600e^{-7x}$ ……….(1)

$\frac{dy}{dx}=500e^{7x}(7)+600e^{-7x}(-7)$

= $500(7)e^{7x}-600(7)e^{7x}$

Now,

$\frac{d^2y}{dx^2}=500(7)e^{7x}(7)-600(7)e^{7x}(-7)$

= $500(49)e^{7x}+600(49)e^{7x}$

=>

$\frac{d^2y}{dx^2}=49\left[500e^{7x}(7)+600e^{7x}\right]$

= $49y$ [Uing equation (1)]

$\Rightarrow$ $\frac{d^2y}{dx^2}$ = $49y$

$\Rightarrow$ Hence proved.

16. If $e^y(x+1)=1$, **show that** $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Solution: Given: $e^y(x+1)=1$

So, $e^y=\frac{1}{x+1}$

Taking log on both the sides, we have

$\log e^{y} = \log \frac{1}{x+1}$

$y \log e = \log 1 - \log(x+1)$

$y = -\log(x+1)$

$\frac{dy}{dx} = -\frac{1}{x+1}\frac{d}{dx}(x+1)$

$= \frac{-1}{x+1} = (x+1)^{-1}$

Again,

$\frac{d^2y}{dx^2} = -(-1)(x+1)^{-2}\frac{d}{dx}(x+1)$

$\left[\because \frac{d}{dx}\{f(x)\}^{n} = n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$

So, $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

Now, L.H.S. = $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

And R.H.S. = $\left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$

L.H.S. = R.H.S.

Hence proved.

17. If $y=\left(\tan^{-1}x\right)^2$, **show that**
$\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2.$

Solution: Given: $y=\left(\tan^{-1}x\right)^2$ ……….(1)

Represent y_1 as first derivative and y_2 as second derivative of the function.

$$y_1=2\left(\tan^{-1}x\right)\frac{d}{dx}\tan^{-1}x$$

$$\left[\because \frac{d}{dx}\{f(x)\}^n=n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$$

and $y_1=2\left(\tan^{-1}x\right)\frac{1}{1+x^2}$

$=\frac{2\tan^{-1}x}{1+x^2}$

So, $\left(1+x^2\right)y_1=2\tan^{-1}x$

Again differentiating both sides with respect to x.

$$\left(1+x^2\right)\frac{d}{dx}y_1+y_1\frac{d}{dx}\left(1+x^2\right)=2.\frac{1}{1+x^2}$$

$$\left(1+x^2\right)y_2+y_1.2x=\frac{2}{1+x^2}$$

$$\left(1+x^2\right)^2 y_2+2xy_1\left(1+x^2\right)=2$$

Hence proved.

Exercise 5.8 Page No: 186

1. Verify Rolle's theorem for $f(x)=x^2+2x-8, x\in[-4,2]$.

Solution: Given function is $f(x)=x^2+2x-8, x\in[-4,2]$

(a) f(x) is a polynomial and polynomial function is always continuous.
So, function is continuous in [-4, 2].

(b) $f'(x)=2x+2,\ f'(x)$ exists in $[-4,2]$, so derivable.

(c) $f(-4)=0$ and $f(2)=0$

$f(-4)=f(2)$

All three conditions of Rolle's theorem are satisfied.

Therefore, there exists, at least one $c\in(-4,2)$ such that $f'(c)=0$

Which implies, 2c + 2 = 0 or c = -1.

2**. Examine if Rolles/ theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's theorem from these examples:**

(i) $f(x)=[x]$ **for** $x\in[5,9]$
(ii) $f(x)=[x]$ **for** $x\in[-2,2]$
(iii) $f(x)=x^2-1$ **for** $x\in[1,2]$

Solution:
(i) Function is greatest integer function.
Given function is not differentiable and continuous
Hence Rolle's theorem is not applicable here.

(ii) Function is greatest integer function.
Given function is not differentiable and continuous.
Hence Rolle's theorem is not applicable.

(iii) $f(x)=x^2-1 \Rightarrow f(1)=(1)^2-1=1-1=0$
$f(2)=(2)^2-1=4-1=3 \therefore f(1)\neq f(2)$

Rolle's theorem is not applicable.

3. If $f:[-5,5]\to$ R is a differentiable function and if f' (x) does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Solution: As per Rolle's theorem, if
(a) f is continuous is $[a,b]$
(b) f is derivable in $[a,b]$
(c) $f(a)=f(b)$
Then, $f'(c)=0, c\in(a,b)$

It is given that f is continuous and derivable, but $f'(c)\neq 0$

$\Rightarrow f(a)\neq f(b)$

$\Rightarrow f(-5)\neq f(5)$

4. Verify Mean Value Theorem if
$f(x)=x^2-4x-3$

in the interval $[a,b]$ where a = 1 and b = 4

Solution:
(a) f(x) is a polynomial.
So, function is continuous in [1, 4] as polynomial function is always continuous.

(b) $f'(x)=2x-4,\ f'(x)$ exists in [1, 4], hence derivable.

Both the conditions of the theorem are satisfied, so there exists, at least one $c\in(1,4)$ such that

$$\frac{f(4)-f(1)}{4-1}=f'(c)$$

$$\frac{-3-(-6)}{3}=2c-4$$

$1=2c-4$

$c=\frac{5}{2}$

5. Verify Mean Value Theorem if $f(x)=x^3-5x^2-3x$ in the interval $[a,b]$ where $a=1$ and $b=3$. Find all $c\in(1,3)$ for which $f'(c)=0$.

Solution:
(a) Function is a polynomial as polynomial function is always continuous.

So continuous in [1, 3]

(b) $f'(x)=3x^2-10x,\ f'(x)$ exists in [1, 3], hence derivable.

Conditions of MVT theorem are satisfied. So, there exists, at least one $c\in(1,3)$ such that

$\frac{f(3)-f(1)}{3-1}=f'(c)$

$\frac{-21-(-7)}{2}=3c^2-10c$

$-7=3c^2-10c$

$3c^2-7c-3c+7=0$

$c(3c-7)-1(3c-7)=0$

$(3c-7)(c-1)=0$

$(3c-7)=0$ or $(c-1)=0$

$3c=7$ or $c=1$

$c=\frac{7}{3}$ or $c=1$

Only $c=\frac{7}{3}\in(1,3)$

As, $f(1) \neq f(3)$, therefore the value of c does not exist such that $f(c)=0$.

6. Examine the applicability of Mean Value Theorem for all the three functions being given below: [Note for students: Check exercise 2]

(i) $f(x)=[x]$ **for** $x \in [5,9]$

(ii) $f(x)=[x]$ **for** $x \in [-2,2]$

(iii) $f(x)=x^2-1$ **for** $x \in [1,2]$

Solution: According to Mean Value Theorem :

For a function $f:[a,b] \to$ R, if

(a) f is continuous on (a,b)

(b) f is differentiable on (a,b)

Then there exist some $c \in (a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

(i) $f(x)=[x]$ for $x \in [5,9]$

given function $f(x)$ is not continuous at $x=5$ and $x=9$.

Therefore,

$f(x)$ is not continuous at $[5,9]$.

Now let n be an integer such that $n \in [5,9]$

$\therefore$ L.H.L. = $\lim\limits_{h\to 0^-}\dfrac{f(n+h)-f(n)}{h} = \lim\limits_{h\to 0^-}\dfrac{(n+h)-(n)}{h} = \lim\limits_{h\to 0^-}\dfrac{n-1-n}{h} = \lim\limits_{h\to 0^-}\dfrac{-1}{h} = \infty$

And R.H.L. = $\lim\limits_{h\to 0^+}\dfrac{f(n+h)-f(n)}{h} = \lim\limits_{h\to 0^+}\dfrac{(n+h)-(n)}{h} = \lim\limits_{h\to 0^+}\dfrac{n-n}{h} = \lim\limits_{h\to 0^+} 0 = 0$

Since, L.H.L. $\neq$ R.H.L.,

Therefore f is not differentiable at $[5,9]$.

Hence Mean Value Theorem is not applicable for this function.

(ii) $f(x)=[x]$ for $x\in[-2,2]$

Given function $f(x)$ is not continuous at $x=-2$ and $x=2$.

Therefore,

$f(x)$ is not continuous at $[-2,2]$.

Now let n be an integer such that $n\in[-2,2]$

$\therefore$ L.H.L. = $\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{(n+h)-(n)}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$

And R.H.L. = $\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{(n+h)-(n)}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$

Since, L.H.L. $\neq$ R.H.L.,

Therefore f is not differentiable at $[-2,2]$.

Hence Mean Value Theorem is not applicable for this function.

(iii) $f(x)=x^2-1$ for $x\in[1,2]$ ……….(1)

Here, $f(x)$ is a polynomial function.

Therefore, $f(x)$ is continuous and derivable on the real line.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

Therefore, both conditions of Mean Value Theorem are satisfied.

Now, From equation (1), we have

$f'(x)=2x$

$f'(c)=2c$

Again, From equation (1):

$$f(a) = f(1) = (1)^2 - 1 = 1 - 1 = 0$$

And, $f(b) = f(2) = (2)^2 - 1 = 4 - 1 = 3$

Therefore,

$$f'c = \frac{f(b) - f(a)}{b - 1}$$

$$2c = \frac{3 - 0}{2 - 1}$$

$$c = \frac{3}{2} \in (1, 2)$$

Therefore, Mean Value Theorem is verified.

Miscellaneous Exercise

Differentiate with respect to x the functions in Exercises 1 to 11.

1. $\left(3x^2-9x+5\right)^9$

Solution: Consider $y=\left(3x^2-9x+5\right)^9$

$$\frac{dy}{dx}=9\left(3x^2-9x+5\right)^8\frac{d}{dx}\left(3x^2-9x+5\right)$$

$$\left[\because \frac{d}{dx}\{f(x)\}^n=n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$$

$$\frac{dy}{dx}=9\left(3x^2-9x+5\right)^8\left[3(2x)-9(1)+0\right]$$

$$\frac{dy}{dx}=27\left(3x^2-9x+5\right)^8\left[2x-3\right]$$

2. $\sin^3 x+\cos^6 x$

Solution: Consider $y=\sin^3 x+\cos^6 x$

or y = $\left(\sin x\right)^3+\left(\cos x\right)^6$

$$\frac{dy}{dx}=3\left(\sin x\right)^2\frac{d}{dx}\sin x+6\left(\cos x\right)^5\frac{d}{dx}\cos x$$

$$\frac{dy}{dx}=3\sin^2 x\cos x-6\cos^5 x\sin x$$

= $3\sin x\cos x\left(\sin x-2\cos^4 x\right)$

3. $\left(5x\right)^{3\cos 2x}$

Solution: Consider $y=\left(5x\right)^{3\cos 2x}$

Taking log both the sides, we get

$$\log y=\log\left(5x\right)^{3\cos 2x}$$

$$\log y = 3\cos 2x \log(5x)$$

Derivate above function:

$$\frac{d}{dx}\log y = 3\frac{d}{dx}\{\cos 2x \log(5x)\}$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{d}{dx}\log(5x) + \log(5x)\frac{d}{dx}\cos 2x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{1}{5x}\frac{d}{dx}5x + \log(5x)(-\sin 2x)\frac{d}{dx}2x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{1}{5x}.5 - 2\sin 2x \log(5x)\right]$$

$$\frac{dy}{dx} = 3y\left[\frac{\cos 2x}{x} - 2\sin 2x \log(5x)\right]$$

$$\frac{dy}{dx} = 3(5x)^{3\cos 2x}\left[\frac{\cos 2x}{x} - 2\sin 2x \log(5x)\right]$$ (using value of y)

4. $\sin^{-1}(x\sqrt{x}), 0 \le x \le 1$

Solution: Consider $y = \sin^{-1}(x\sqrt{x})$

or y = $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}\frac{d}{dx}x^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{1-x^3}}.\frac{3}{2}x^{\frac{1}{2}}$$

= $\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$

5. $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

Solution: Consider $y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Apply derivation:

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\frac{d}{dx}\cos^{-1}\frac{x}{2} - \cos^{-1}\frac{x}{2}\frac{d}{dx}\sqrt{2x+7}}{\left(\sqrt{2x+7}\right)^2}$$

[Using Quotient Rule]

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\frac{d}{dx}\frac{x}{2} - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2}(2x+7)^{\frac{-1}{2}}\frac{d}{dx}(2x+7)}{\left(\sqrt{2x+7}\right)^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{2x+7}.\frac{2}{\sqrt{4-x^2}}.\frac{1}{2} - \frac{1}{2}.\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}}{(2x+7)}$$

$$= \frac{-\left[\frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right]}{(2x+7)}$$

$$\frac{dy}{dx}=-\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\sqrt{2x+7}\left(2x+7\right)}\right]$$

$$=-\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\left(2x+7\right)^{\frac{3}{2}}}\right]$$

6. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0<x<\frac{\pi}{2}$

Solution: Consider $y=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0<x<\frac{\pi}{2}$ ……….(i)

Reduce the functions into simplest form,

$$\sqrt{1+\sin x}=\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}=\cos\frac{x}{2}+\sin\frac{x}{2}$$

And $\sqrt{1-\sin x}=\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$=\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}=\cos\frac{x}{2}-\sin\frac{x}{2}$$

Now, we are available with the equation below:

$$y=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$= \cot^{-1}\left(\dfrac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$

$y = \cot^{-1}\left(\cot\dfrac{x}{2}\right)$

$= \dfrac{x}{2}$

Apply derivation:

$\dfrac{dy}{dx} = \dfrac{1}{2}(1) = \dfrac{1}{2}$

7. $(\log x)^{\log x}, x > 1$

Solution: Consider $y = (\log x)^{\log x}, x > 1$ ……….(i)

Taking log both sides:

$\log y = \log(\log x)^{\log x} = \log x \log(\log x)$

Apply derivation:

$\dfrac{d}{dx}(\log y) = \dfrac{d}{dx}(\log x \log(\log x))$

$\dfrac{1}{y}\dfrac{dy}{dx} = \log x \dfrac{d}{dx}\log(\log x) + \log(\log x)\dfrac{d}{dx}\log x$

$\dfrac{1}{y}\dfrac{dy}{dx} = \log x \dfrac{1}{\log x}\dfrac{d}{dx}(\log x) + \log(\log x)\dfrac{1}{x}$

$= \dfrac{1}{x} + \dfrac{\log(\log x)}{x}$

$\dfrac{dy}{dx} = y\left(\dfrac{1+\log(\log x)}{x}\right)$

$= (\log x)^{\log x}\left(\frac{1+\log(\log x)}{x}\right)$

8. $\cos(a\cos x + b\sin x)$ **for some constants a and b.**

Solution: Consider $y=\cos(a\cos x+b\sin x)$ for some constants a and b.
Apply derivation:

$$\frac{dy}{dx} = -\sin(a\cos x+b\sin x)\frac{d}{dx}(a\cos x+b\sin x)$$

$$\frac{dy}{dx} = -\sin(a\cos x+b\sin x)(-a\sin x+b\cos x)$$

$$\frac{dy}{dx} = -(-a\sin x+b\cos x)\sin(a\cos x+b\sin x)$$

$$\frac{dy}{dx} = (-a\sin x-b\cos x)\sin(a\cos x+b\sin x)$$

9. $(\sin x-\cos x)^{\sin x-\cos x}, \frac{\pi}{4}<x<\frac{3\pi}{4}$

Solution: Consider $y=(\sin x-\cos x)^{\sin x-\cos x}$ ……….(i)
Apply log both sides:

$\log y = \log(\sin x-\cos x)^{\sin x-\cos x}$

$= (\sin x-\cos x)\log(\sin x-\cos x)$

Apply derivation:

$$\frac{d}{dx}\log y = (\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)+\log(\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = (\sin x-\cos x)\frac{1}{(\sin x-\cos x)}\frac{d}{dx}(\sin x-\cos x)+\log(\sin x-\cos x).(\cos x+\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = (\cos x+\sin x)+(\cos x+\sin x)\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

$$\frac{dy}{dx}=y(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

$$\frac{dy}{dx}=(\sin x-\cos x)^{\sin x-\cos x}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

10. $x^x+x^a+a^x+a^a$, **for some fixed a> 0 and x>0.**

Solution: Consider $y=x^x+x^a+a^x+a^a$

Apply derivation:

$$\frac{dy}{dx}=\frac{d}{dx}x^x+\frac{d}{dx}x^a+\frac{d}{dx}a^x+\frac{d}{dx}a^a$$

$$=\frac{d}{dx}x^x+ax^{a-1}+a^x\log a+0 \quad ....(i)$$

First term from equation (i) :

$\frac{d}{dx}(x^x)$, Consider $u=x^x$

$\log u=\log x^x = x\log x$

$$\frac{d}{dx}\log u=\frac{d}{dx}(x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\log x+\log x\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{x}+\log x.1$$

$$=1+\log x$$

This implies,

$$\frac{du}{dx}=u(1+\log x)$$

Substitute value of u back:

$\frac{d}{dx}x^x = x^x(1+\log x)$ …(ii)

Using equation (ii) in (i), we have

$\frac{dy}{dx} = x^x(1+\log x)ax^{a-1} + a^x \log a$

11. $x^{x^2-3} + (x-3)^{x^2}$ **for x>3.**

Solution: Consider $y = x^{x^2-3} + (x-3)^{x^2}$ for x>3.

Put $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ……….(i)

Now $u = x^{x^2-3}$

$\log u = \log x^{x^2-3}$ = $(x^2-3)\log x$

$\frac{1}{u}\frac{du}{dx} = (x^2-3)\frac{d}{dx}\log x + \log x \frac{d}{dx}(x^2-3)$

= $(x^2-3)\frac{1}{x} + \log x(2x-0)$

$\frac{1}{u}\frac{du}{dx} = \frac{x^2-3}{x} + 2x\log x$

$\frac{du}{dx} = u\left(\frac{x^2-3}{x} + 2x\log x\right)$

$\frac{du}{dx} = x^{x^2-3}\left(\frac{x^2-3}{x} + 2x\log x\right)$ ……….(ii)

Again $v=(x-3)^{x^2}$

$\log v=\log(x-3)^{x^2}$

$= x^2\log(x-3)$

$\frac{1}{v}\frac{dv}{dx}=x^2\frac{d}{dx}\log(x-3)+\log(x-3)\frac{d}{dx}x^2$

$= x^2\frac{1}{x-3}\frac{d}{dx}(x-3)+\log(x-3)2x$

$\frac{1}{v}\frac{dv}{dx}=\frac{x^2}{x-3}+2x\log(x-3)$

$\frac{dv}{dx}=v\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$

$\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$ ....(iii)

Using equation (ii) and (iii) in eq. (i), we have

$\frac{dy}{dx}=x^{x^2-3}\left(\frac{x^2-3}{x}+2x\log x\right)+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$

12. Find $\frac{dy}{dx}$ **if** $y=12(1-\cos t)$ **and** $x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$.

Solution: Given expressions are $y=12(1-\cos t)$ and $x=10(t-\sin t)$

$\frac{dy}{dt}=12\frac{d}{dt}(1-\cos t)=12(0+\sin t)=12\sin t$

and $\frac{dx}{dt}=10\frac{d}{dt}(1-\cos t)$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{12\sin t}{10(1-\cos t)}$

$= \frac{6}{5} \cdot \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}$

$= \frac{6}{5} \cdot \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$

13. Find $\frac{dy}{dx}$ **if** $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1.$

Solution: Given expression is $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$
Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \frac{d}{dx}\sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \frac{1}{2}\left(1-x^2\right)^{\frac{-1}{2}} \frac{d}{dx}\left(1-x^2\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-1+x^2}} \frac{1}{2\sqrt{1-x^2}}(-2x)$$

$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right)$

Which implies:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}}$$

$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

Therefore, dy/dx = 0

14.

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, **for** $-1<x<1$,

Prove that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$.

Solution: Given expression is $x\sqrt{1+y}+y\sqrt{1+x}=0$

$x\sqrt{1+y}=-y\sqrt{1+x}$

Squaring both sides:

$x^2(1+y)=y^2(1+x)$

$x^2+x^2y=y^2+y^2x$

$x^2-y^2=-x^2y+y^2x$

$(x-y)(x+y)=-xy(x-y)$

$x+y=-xy$

=> $y(1+x)=-x$

=> $y=\frac{-x}{1+x}$

Apply derivation:

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}x-x\frac{d}{dx}(1+x)}{(1+x)^2}$$

= $-\frac{(1+x).1-x.1}{(1+x)^2}$

= $-\frac{1}{(1+x)^2}$

Hence Proved.

15.

If $(x-a)^2+(y-b)^2=c^2,$ **for some c>0, prove that**

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

Solution: Given expression is $(x-a)^2+(y-b)=c^2$ ……….(1)
Apply derivation:

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$2(x-a)=-2(y-b)\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\left(\frac{x-a}{y-b}\right)$$ ……….(2)

Again $$\frac{d^2y}{dx^2}=\frac{-\left[(y-b).1-(x-a)\frac{dy}{dx}\right]}{(y-b)^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\left[(y-b).1-(x-a)\left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2}$$ [Using equation (2)]

$$\frac{d^2y}{dx^2}=\frac{-\left[(y-b)+\left(\frac{(x-a)^2}{y-b}\right)\right]}{(y-b)^2}$$

$$=\frac{-\left[(y-b)^2+(x-a)^2\right]}{(y-b)^3}$$

$$=\frac{-c^2}{(y-b)^3}$$ ……….(3)

Put values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given, we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[(y-b)^2+(x-a)^2\right]^{\frac{3}{2}}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} = \frac{(c^2)^{\frac{3}{2}}}{-c^2} = -c$$ (Constant value)

Which is a constant and is independent of a and b.

16. If $\cos y = x\cos(a+y)$ **with** $\cos a \neq \pm 1$, **prove that** $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Solution: Given expression is $\cos y = x\cos(a+y)$

$$x = \frac{\cos y}{\cos(a+y)}$$

Apply derivative w.r.t. y

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)$$

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y\{-\sin(a+y)\}}{\cos^2(a+y)}$$

$= \dfrac{-\cos(a+y)\sin y+\sin(a+y)\cos y}{\cos^2(a+y)}$

$\Rightarrow \dfrac{dx}{dy}=\dfrac{\sin(a+y-y)}{\cos^2(a+y)}$

$= \dfrac{\sin a}{\cos^2(a+y)}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$ [Take reciprocal]

17. If $x=a(\cos t+t\sin t)$ **and** $y=a(\sin t-t\cos t)$, **find** $\dfrac{d^2y}{dx^2}$.

Solution: Given expressions are $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

$x=a(\cos t+t\sin t)$

Differentiating both sides w.r.t. t

$$\frac{dx}{dt}=a\left(-\sin t+\frac{d}{dt}t\sin t\right)$$

$$\frac{dx}{dt}=a\left(-\sin t+t\frac{d}{dt}\sin t+\sin t\frac{d}{dt}t\right)$$

$$\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)$$

$$\Rightarrow \frac{dx}{dt}=at\cos t$$

And:

$y = a(\sin t - t\cos t)$

Differentiating both sides w.r.t. t

$\frac{dy}{dt} = a\left(\cos t - \frac{d}{dt} t\cos t\right)$

$\frac{dy}{dt} = a\left(\cos t - \left(t\frac{d}{dt}\cos t + \cos t\frac{d}{dt}t\right)\right)$

$\frac{dy}{dt} = a(\cos t - (-t\sin t + \cos t))$

$\frac{dy}{dt} = at\sin t$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \frac{\sin t}{\cos t} = \tan t$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\tan t$ = $\sec^2 t\frac{d}{dx}t$

= $\sec^2 t\frac{dt}{dx}$ = $\sec^2 t\frac{1}{at\cos t}$

= $\sec^2 t\frac{\sec t}{at} = \frac{\sec^3 t}{at}$

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Solution: Given expression is $f(x) = |x^3|$ = $\begin{cases} x^3, & \text{if } x \ge 0 \\ (-x^3), & \text{if } x < 0 \end{cases}$

Step 1: when x < 0

f(x) = -x^3

Differentiate w.r.t. to x,

f'(x) = -3x^2

Differentiate w.r.t. to x,

f''(x) = -6x, exist for all values of x < 0.

Step 2: When x ≥ 0

f(x) = x^3

Differentiate w.r.t. to x,

f'(x) = 3x^2

Differentiate w.r.t. to x,

f''(x) = 6x, exist for all values of x > 0.

Step 3: When x = 0

$$\lim_{h\to 0^-} \frac{f(0)-f(0+h)}{h} = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = f'(c)$$

$$f'(x) = \begin{cases} 3x^2, & \text{if } \quad x \geq 0 \\ -3x^2, & \text{if } \quad x < 0 \end{cases}$$

Now, Check differentiability at x = 0

L.H.D. at x = 0

$$\lim_{h\to 0^-} \frac{f\prime(0)-f\prime(0+h)}{h}$$

$$= \lim_{h\to 0^-} \frac{3(0)-(-3(-h)^2)}{h}$$

$$= \lim_{h\to 0^-} \frac{3h^2}{h}$$

As h = 0,

= 0

And R.H.D. at x = 0

$$\lim_{h\to 0^+} \frac{f'(0+h)-f'(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{f\prime(h)-f\prime(0)}{h}$$

= $\lim_{h\to 0^+} \frac{3(h)^2 - 3(0)^2}{h}$

= $\lim_{h\to 0^+}$ 3h = 0 (at h = 0)

Again L.H.D. at $x = 0$ = R.H.D. at x = 0.

This implies, f''(x) exists and differentiable at all real values of x.

19. Using mathematical induction, prove that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for all positive integers n.

Solution: Consider p(n) be the given statement.

$p(n) = \frac{d}{dx}\left(x^n\right) = nx^{n-1}$ .....(1)

Step 1: Result is true at n = 1

$p(1) = \frac{d}{dx}\left(x^1\right) = (1)x^{1-1}$ = $(1)x^0 = 1$,

which is true as $\frac{d}{dx}(x) = 1$

Step 2: Suppose $p(m)$ is true.

$p(m) = \frac{d}{dx}\left(x^m\right) = mx^{m-1}$ ....(2)

Step 3: Prove that result is true for n = m+1.

$p(m+1) = \frac{d}{dx}\left(x^{m+1}\right) = (m+1)x^{m+1-1}$

$x^{m+1} = x^1 + x^m$

$\frac{d}{dx}x^{m+1} = \frac{d}{dx}\left(x.x^m\right)$

= $x.\frac{d}{dx}x^m + x^m\frac{d}{dx}x$

= $x.mx^{m-1} + x^m(1)$

Therefore, $mx^{m}+x^{m}=x^{m}(m+1)$

$(m+1)x^{m}=(m+1)x^{m}$

$(m+1)x^{(m+1)-1}$

Therefore, $p(m+1)$ is true if $p(m)$ is true but $p(1)$ is true.

Thus, by Principal of Induction p(n) is true for all n ∈ N.

20. Using the fact that $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and the differentiation, obtain the sum formula for cosines.

Solution: Given expression is $\sin(A+B)=\sin A\cos B+\cos A\sin B$
Consider A and B as function of t and differentiating both sides w.r.t. x,

$$\cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)=\sin A(-\sin B)\frac{dB}{dt}+\cos B\left(\cos A\frac{dA}{dt}\right)+\cos A\cos B\frac{dB}{dt}+\sin B(-\sin A)\frac{dA}{dt}$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)=(\cos A\cos B-\sin A\sin B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)$$

$$\Rightarrow \cos(A+B)=(\cos A\cos B-\sin A\sin B)$$

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points?

Solution: Consider us consider the function $f(x)=|x|+|x-1|$
f is continuous everywhere but it is not differentiable at x = 0 and x = 1.

22. If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, **prove that**

$$\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

Solution: Given expression is

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Apply derivative:

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}f(x) & \frac{d}{dx}g(x) & \frac{d}{dx}h(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

23. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1,$ **show that** $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$

Solution: Given expression is $y = e^{a\cos^{-1}x}$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}.\frac{d}{dx}a\cos^{-1}x$$

$$= e^{a\cos^{-1}x}.a\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

This implies,

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating both sides with respect to x, we have

$$\left(1-x^2\right)2.\frac{dy}{dx}.\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-2x)=2a^2y\frac{dy}{dx}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=a^2y$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-a^2y=0$$

Hence Proved.