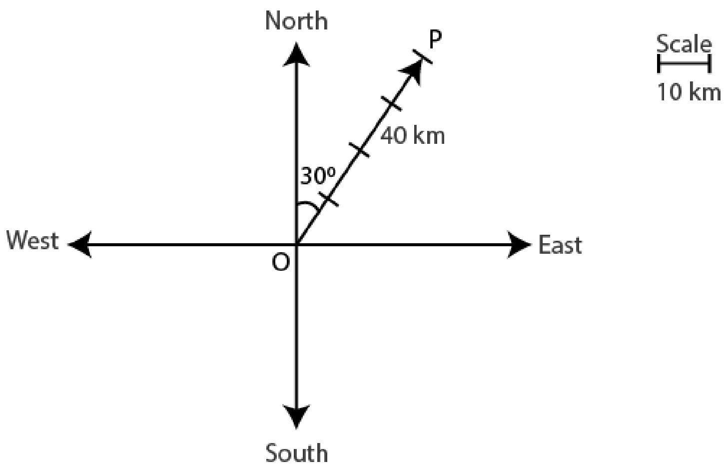
Exercise 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.

Solution:



The vector $\overline{OP}$ represents the displacement of 40 km, 30° east of north.

2. Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 metres north-west (iii) 40°

(iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2

Solution:

(i) 10 kg is a scalar quantity because it has only magnitude.
(ii) 2 meters north-west is a vector quantity as it has both magnitude and direction.
(iii) 40° is a scalar quantity as it has only magnitude.
(iv) 40 watts is a scalar quantity as it has only magnitude.
(v) 10^{-19} coulomb is a scalar quantity as it has only magnitude.
(vi) 20 m/s^2 is a vector quantity as it has both magnitude and direction.

3. Classify the following as scalar and vector quantities.

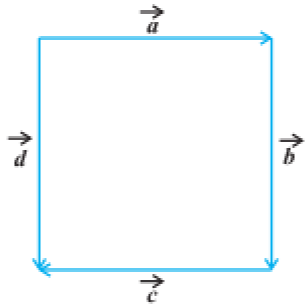
(i) time period (ii) distance (iii) force

(iv) velocity (v) work done

Solution:

(i) Time period is a scalar quantity as it has only magnitude.
(ii) Distance is a scalar quantity as it has only magnitude.
(iii) Force is a vector quantity as it has both magnitude and direction.
(iv) Velocity is a vector quantity as it has both magnitude as well as direction.
(v) Work done is a scalar quantity as it has only magnitude.

4. In Figure, identify the following vectors.



(i) Coinitial (ii) Equal (iii) Collinear but not equal
Solution:

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial because they have the same initial point.

(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal because they have the same magnitude and direction.

(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, their directions are not the same.

5. Answer the following as true or false.
(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.
Solution:

(i) True.
Vectors $\vec{a}$ and $-\vec{a}$ are parallel to the same line.
(ii) False.
Collinear vectors are those vectors that are parallel to the same line.
(iii) False.
Two vectors having the same magnitude need not necessarily be parallel to the same line.
(iv) False.
Only if the magnitude and direction of two vectors are the same, regardless of the positions of their initial points the two vector are said to be equal.

Exercise 10.2

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Solution:

Given vectors are:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2}$$

$$= \sqrt{4 + 49 + 9}$$

$$= \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

2. Write two different vectors having same magnitude.
Solution:

Consider $\vec{a} = \left(\hat{i} - 2\hat{j} + 4\hat{k}\right)$ and $\vec{b} = \left(2\hat{i} + \hat{j} - 4\hat{k}\right)$.

It can be observed that $|\vec{a}| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$ and

$|\vec{b}| = \sqrt{2^2 + 1^2 + (-4)^2} = \sqrt{4 + 1 + 16} = \sqrt{21}$

Thus, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. Here, the vectors are different as they have different directions.

3. Write two different vectors having same direction.
Solution:
Two different vectors having same directions are:
Let us

Consider $\vec{p} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ and $\vec{q} = \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}, \quad m = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}, \text{ and } n = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of $\vec{q}$ are given by

$$l = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},\ m = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

and $n = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$.

4. Find the values of *x* and *y* so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal

Solution:

Given vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ will be equal only if their corresponding components are equal.
Thus, the required values of *x* and *y* are 2 and 3 respectively.

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Solution:

The scalar and vector components are:
The vector with initial point P (2, 1) and terminal point Q (–5, 7) can be shown as,

$$\overrightarrow{PQ} = (-5-2)\hat{i} + (7-1)\hat{j}$$
$$\overrightarrow{PQ} = -7\hat{i} + 6\hat{j}$$

Thus, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}$ and $6\hat{j}$.

6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$

Solution:

Let us find the sum of the vectors:

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$

Hence,

$$\vec{a}+\vec{b}+\vec{c} = (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k}$$
$$= 0\cdot\hat{i} - 4\hat{j} - 1\cdot\hat{k}$$
$$= -4\hat{j} - \hat{k}$$

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

Solution:

We know that

The unit vector $\hat{a}$ in the direction of vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

So,

$$|\vec{a}| = \sqrt{1^2+1^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

Thus,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively

Solution:

We know that,

Given points are P (1, 2, 3) and Q (4, 5, 6).

So, $\overrightarrow{PQ} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$

Thus, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$

Solution:

We know that,

Given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\therefore \vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} = 1\hat{i} + 0\hat{j} + 1\hat{k} = \hat{i} + \hat{k}$

$|\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Thus, the unit vector in the direction of $(\vec{a} + \vec{b})$ is

$$\frac{\left(\vec{a} + \vec{b}\right)}{\left|\vec{a} + \vec{b}\right|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

Solution:

Firstly,

Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

So,

$|\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25+1+4} = \sqrt{30}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Thus, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by,

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$
$$= 8\left(\frac{5\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{30}}\right)$$
$$= \frac{40}{\sqrt{30}}\vec{i} - \frac{8}{\sqrt{30}}\vec{j} + \frac{16}{\sqrt{30}}\vec{k}$$

11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Solution:

Firstly,

Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

It is seen that $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2\left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = -2\vec{a}$

$\therefore \vec{b} = \lambda\vec{a}$

where,

$\lambda = -2$

Therefore, we can say that the given vectors are collinear.

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

Firstly,

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

The modulus is given by,

$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

Thus, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

13. Find the direction cosines of the vector joining the points A (1, 2, –3) and B (–1, –2, 1) directed from A to B.

Solution:

We know that the

Given points are A (1, 2, –3) and B (–1, –2, 1).

Now,

$\overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$

$\overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

Therefore, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Solution:

Firstly,

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Then,

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

So, we have $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Therefore, the given vector is equally inclined to axes OX, OY, and OZ.

15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2:1

(i) internally

(ii) externally

Solution:

We know that

The position vector of point R dividing the line segment joining two points P and Q in the ratio *m: n* is given by:

(i) Internally: $\frac{m\vec{b} + n\vec{a}}{m+n}$

(ii) Externally: $\frac{m\vec{b} - n\vec{a}}{m-n}$

$\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} = (-2\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})$$

$$= -3\hat{i} + 3\hat{k}$$

16. Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).

Solution:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by,

$$\overline{OR} = \frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2} = \frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$
$$= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2} = 3\hat{i}+2\hat{j}+\hat{k}$$

17. Show that the points A, B and C with position vectors, $\vec{a}=3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}-5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution:

We know

Given position vectors of points A, B, and C are:

$\vec{a}=3\hat{i}-4\hat{j}-4\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}-5\hat{k}$

$\therefore \overrightarrow{AB}=\vec{b}-\vec{a}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$

$\overrightarrow{BC}=\vec{c}-\vec{b}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{CA}=\vec{a}-\vec{c}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

Now,

$\left|\overrightarrow{AB}\right|^2=(-1)^2+3^2+5^2=1+9+25=35$

$\left|\overrightarrow{BC}\right|^2=(-1)^2+(-2)^2+(-6)^2=1+4+36=41$

$\left|\overrightarrow{CA}\right|^2=2^2+(-1)^2+1^2=4+1+1=6$

Hence,

$$\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{CA}\right|^2=35+6=41=\left|\overrightarrow{BC}\right|^2$$

Hence, proved that the given points form the vertices of a right angled triangle.

18. In triangle ABC (Fig 10.18) which of the following is not true:

(A) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$

(B) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

(C) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

(D) $\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

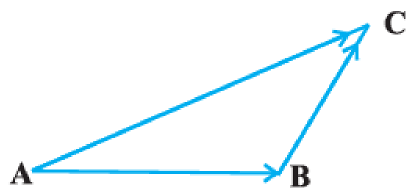


Fig 10.18

Solution:

Firstly let us consider,

Applying the triangle law of addition in the given triangle, we get:

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ...(1)

$\overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ ...(2)

$\therefore$ The equation given in alternative A is true.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

$\therefore$ The equation given in alternative B is true.

From equation (2), we have:

$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

$\therefore$ The equation given in alternative D is true.

Now, consider the equation given in alternative C:

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$ ...(3)

From equations (1) and (3), we get:

$\overrightarrow{AC}=\overrightarrow{CA}$

$\overrightarrow{AC}=-\overrightarrow{AC}$

$\overrightarrow{AC}+\overrightarrow{AC}=\vec{0}$

$2\overrightarrow{AC}=\vec{0}$

$\overrightarrow{AC}=\vec{0}$, which is not true.

Thus, the equation given in alternative C is incorrect.

The correct answer is **C.**

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

A. $\vec{b}=\lambda\vec{a}$, for some scalar λ

B. $\vec{a}=\pm\vec{b}$

C. the respective components of $\vec{a}$ and $\vec{b}$ are proportional

D. both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

We know

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.

So, we have:

$\vec{b}=\lambda\vec{a}$ (For some scalar λ)

If $\lambda=\pm1$, then $\vec{a}=\pm\vec{b}$.

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\vec{b} = \lambda\vec{a}.$$
$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$
$$b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$
$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Hence, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

But, vectors $\vec{a}$ and $\vec{b}$ can have different directions.

Thus, the statement given in **D** is **incorrect**.

The correct answer is **D**.

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes √3 and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.

Solution:

Firstly let us consider,

$$|\vec{a}|=\sqrt{3},\ |\vec{b}|=2 \text{ and, } \vec{a}\cdot\vec{b}=\sqrt{6}$$

Now, we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$$

$$\cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$\cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Thus, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$

2. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

Solution:

Let us consider the

Given vectors are: $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$

$$\text{Now, } \vec{a}\cdot\vec{b}=(\hat{i}-2\hat{j}+3\hat{k})(3\hat{i}-2\hat{j}+\hat{k})$$

$$=1.3+(-2)(-2)+3.1$$

$$=3+4+3$$

$$=10$$

Also, we know that $\vec{a}\cdot b=|\vec{a}||b|\cos\theta$.

$$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$$

$$\cos\theta=\frac{10}{14}$$

$$\theta=\cos^{-1}\left(\frac{5}{7}\right)$$

Hence, the angle between the vectors is $\cos^{-1}$ (5/7).

3. Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

Solution:
Firstly,
Let $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}$.
Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,
$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b})=\frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\}=\frac{1}{\sqrt{2}}(1-1)=0$$
Thus, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
Solution:
Firstly,
Let $\vec{a}=\hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b}=7\hat{i}-\hat{j}+8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$
Hence, the projection is 60/√114.

5. Show that each of the given three vectors is a unit vector:
$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$
Also, show that they are mutually perpendicular to each other.
Solution:
It is given that
$$\text{Let } \vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k},$$
$$\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$$
$$\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$$
$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$
$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$
$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$
Hence, each of the given three vectors is a unit vector.

$$\vec{a}\cdot\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}\cdot\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}\cdot\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Therefore, the given threee vectors are mutually perpendicular to each other.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$

Solution:

Let us consider,

$$(\vec{a}\cdot\vec{b}).(\vec{a}-\vec{b})=8$$

$$\vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$

$$|\vec{a}|^2-|\vec{b}|^2=8$$

$$\left(8|\vec{b}|\right)^2-|\vec{b}|^2=8 \qquad \left[|\vec{a}|=8|\vec{b}|\right]$$

$$64|\vec{b}|^2-|\vec{b}|^2=8$$

$$63|\vec{b}|^2=8$$

$$|\vec{b}|^2=\frac{8}{63}$$

$$|\vec{b}|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$

$$|\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

And,

$$|\vec{a}|=8|\vec{b}|=\frac{8\times2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

7. Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$

Solution:

Let us consider the given expression,

$$\left(3\vec{a}-5\vec{b}\right)\cdot\left(2\vec{a}+7\vec{b}\right)$$
$$=3\vec{a}\cdot 2\vec{a}+3\vec{a}\cdot 7\vec{b}-5\vec{b}\cdot 2\vec{a}-5\vec{b}\cdot 7\vec{b}$$
$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$
$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is ½.
Solution:
Firstly,
Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.
It is given that $|\vec{a}|=|\vec{b}|,\ \vec{a}\cdot\vec{b}=\frac{1}{2}$, and $\theta=60°$. ...(1)
We know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.
$$\therefore\frac{1}{2}=|\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using }(1)]$$
$$\frac{1}{2}=|\vec{a}|^2\times\frac{1}{2}$$
$$|\vec{a}|^2=1$$
$$|\vec{a}|=|\vec{b}|=1$$
Hence the magnitude of two vectors is 1.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a},\ (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$
Solution:
Let us consider,
$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$$
$$\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=12$$
$$|\vec{x}|^2-|\vec{a}|^2=12$$
$$|\vec{x}|^2-1=12 \qquad [|\vec{a}|=1\text{ as }\vec{a}\text{ is a unit vector}]$$
$$|\vec{x}|^2=13$$
$$\therefore|\vec{x}|=\sqrt{13}$$
Hence the value is √13.

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.
Solution:

We know that the

Given vectors are $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, and $\vec{c}=3\hat{i}+\hat{j}$.

Now,

$\vec{a}+\lambda\vec{b}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$

If $\left(\vec{a}+\lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$\left(\vec{a}+\lambda\vec{b}\right)\cdot\vec{c}=0.$

$\left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right]\cdot\left(3\hat{i}+\hat{j}\right)=0$

$(2-\lambda)3+(2+2\lambda)1+(3+\lambda)0=0$

$6-3\lambda+2+2\lambda=0$

$-\lambda+8=0$

$\lambda=8$

Therefore, the required value of λ is 8.

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution:

Let us consider,

$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$=|\vec{a}|^2\,\vec{b}\cdot\vec{b}-|\vec{a}||\vec{b}|\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|\vec{a}\cdot\vec{b}-|\vec{b}|^2\,\vec{a}\cdot\vec{a}$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$

$=0$

Therefore, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are perpendicular to each other.

12. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

Solution:

We know

Given, $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$.

Now,

$\vec{a}\cdot\vec{a}=0\Rightarrow|\vec{a}|^2=0\Rightarrow|\vec{a}|=0$

$\therefore\vec{a}$ is a zero vector.

Thus, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b}=0$ can be any vector.

13. If $\vec{a},\ \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

Solution:

Consider the given vectors,

Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

So,

$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{a}\cdot\vec{0}$

$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=\vec{a}\cdot\vec{0}$ [Distributivity of scalar product over addition]

$1+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0 \quad \dots (1)$ $\left[\begin{array}{l}\vec{a}\cdot\vec{a}=|\vec{a}|\cdot|\vec{a}|\cos 0°=1 \\ (\vec{a} \text{ is unit vector} \Rightarrow |\vec{a}|=1)\end{array}\right]$

Next,

$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{b}\cdot\vec{0}$

$\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}=\vec{b}\cdot\vec{0}$

$\vec{b}\cdot\vec{a}+1+\vec{b}\cdot\vec{c}=0 \quad \dots (2)$ $[\vec{b}\cdot\vec{b}=1]$

And,

$\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{c}\cdot\vec{0}$

$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=\vec{c}\cdot\vec{0}$

$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+1=0 \quad \dots (3)$ $[\vec{c}\cdot\vec{c}=1]$

From (1), (2) and (3),

$(1+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})+(\vec{b}\cdot\vec{a}+1+\vec{b}\cdot\vec{c})+(\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+1)=0+0+0$

$(3+\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a})+(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c})+(\vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c})=0$ [Scalar product is commutative]

$3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-3}{2}$

Hence the value is -3/2.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Solution:

Firstly,

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then, their dot product is given by:

$\vec{a}\cdot\vec{b}=2.3+4.3+3(-6)=6+12-18=0$

Now, it's seen that

$|\vec{a}|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$

$\therefore \vec{b}\neq\vec{0}$

Therefore, the converse of the given statement need not be true.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Solution:

We know

The vertices of ΔABC are given as A (1, 2, 3), B (–1, 0, 0), and C (0, 1, 2).

Also given, ∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 2\times1 + 2\times1 + 3\times2 = 2+2+6 = 10$$

$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, we know that

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right| \cos(\angle ABC).$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6} \cos(\angle ABC)$$

$$\cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$

$$\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Hence, the angle is $\cos^{-1}$ (10/ √102).

16. Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear.

Solution:

Let us consider

Given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, –1).

Now,

$$\overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

Now,

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Therefore, the given points A, B, and C are collinear.

17. Show that the vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ **form the vertices of a right angled triangle.**

Solution:

Firstly consider,

Let vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ be position vectors of points A, B, and C respectively.

So,

$\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Now, vectors $\overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

Hence,

$$\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$$

$$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$$

$$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\therefore \left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2=6+35=41=\left|\overrightarrow{AB}\right|^2$$

Therefore, ΔABC is a right-angled triangle.

18. If $\vec{a}$ **is a nonzero vector of magnitude '*a*' and λ a nonzero scalar, then** $\lambda\vec{a}$ **is unit vector if**

(A) λ = 1 (B) λ = – 1 (C) *a* = |λ| (D) *a* = 1/|λ|

Solution:

Explanation:

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$$|\lambda\vec{a}|=1$$

$$|\lambda||\vec{a}|=1$$

$$|\vec{a}|=\frac{1}{|\lambda|} \qquad [\lambda\neq 0]$$

$$a=\frac{1}{|\lambda|} \qquad \left[|\vec{a}|=a\right]$$

Therefore, vector $\lambda\vec{a}$ is a unit vector if $a=\frac{1}{|\lambda|}$

The correct answer is D.

Exercise 10.4

1. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

It is given that,

$\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

Therefore,

$$|\vec{a}\times\vec{b}|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Solution:

It is given that,

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

So, we have

$\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\ \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

Thus,

$$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|=\sqrt{16^2+(-16)^2+(-8)^2}$$

$$=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}=8\times3=24$$

Therefore, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$=\pm\frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

3. If a unit vector $\vec{a}$ makes an angles $\frac{\pi}{3}$ with $\hat{i},\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$.

Solution:

Firstly,

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

As $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also given, that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1 \qquad [|\vec{a}| = 1]$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad [|\vec{a}| = 1]$$

Also, $\cos\theta = \frac{a_3}{|\vec{a}|}$.

$\Rightarrow a_3 = \cos\theta$

Now,

$$|a| = 1$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\frac{3}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Thus, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

4. **Show that**

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$$

Solution:

Firstly consider the LHS,
We have,

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$$
$$=(\vec{a}-\vec{b})\times\vec{a}+(\vec{a}-\vec{b})\times\vec{b} \quad \text{[By distributivity of vector product over addition]}$$
$$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b} \quad \text{[Again, by distributivity of vector product over addition]}$$
$$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$$
$$=2(\vec{a}\times\vec{b})$$

5. Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$.
Solution:
It is given that,
Given,

$$(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$
$$\hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have

$$6\mu-27\lambda=0$$
$$2\mu-27=0$$
$$2\lambda-6=0$$

Now,

$$2\lambda-6=0 \Rightarrow \lambda=3$$
$$2\mu-27=0 \Rightarrow \mu=\frac{27}{2}$$

Thus, $\lambda=3$ and $\mu=\frac{27}{2}$.

6. Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?
Solution:
It is given that,

$\vec{a}\cdot\vec{b}=0$

Then,

(i) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\perp\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$\vec{a}\times\vec{b}=0$

(ii) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\parallel\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Therefore, $|\vec{a}|=0$ or $|\vec{b}|=0$.

7. Let the vectors $\vec{a},\ \vec{b},\ \vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

It is given that,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$

$$(\vec{b}+\vec{c})=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}$$

$$\text{Now, }\vec{a}\times(\vec{b}+\vec{c})\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\hat{i}[a_2(b_3+c_3)-a_3(b_2+c_2)]-\hat{j}[a_1(b_3+c_3)-a_3(b_1+c_1)]+\hat{k}[a_1(b_2+c_2)-a_2(b_1+c_1)]$$

$$=\hat{i}[a_2b_3+a_2c_3-a_3b_2-a_3c_2]+\hat{j}[-a_1b_3-a_1c_3+a_3b_1+a_3c_1]+\hat{k}[a_1b_2+a_1c_2-a_2b_1-a_2c_1]\quad \ldots(1)$$

And,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\hat{i}[a_2b_3-a_3b_2]+\hat{j}[b_1a_3-a_1b_3]+\hat{k}[a_1b_2-a_2b_1]\qquad(2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}[a_2c_3-a_3c_2]+\hat{j}[a_3c_1-a_1c_3]+\hat{k}[a_1c_2-a_2c_1]\qquad(3)$$

On adding (2) and (3), we get

$$(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})=\hat{i}[a_2b_3+a_2c_3-a_3b_2-a_3c_2]+\hat{j}[b_1a_3+a_3c_1-a_1b_3-a_1c_3]+\hat{k}[a_1b_2+a_1c_2-a_2b_1-a_2c_1]\qquad(4)$$

From (1) and (4), we obtain

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

- Hence proved.

8. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example.

Solution:

Firstly let us consider,

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&4\\4&6&8\end{vmatrix}=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

Now, it's seen that

$$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$$\therefore \vec{a}\neq\vec{0}$$

$$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\therefore \vec{b}\neq\vec{0}$$

Thus, the converse of the given statement need not be true.

9. Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Solution:

We know,

Given A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5) are the vertices of triangle ABC.

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$$\overrightarrow{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$$

$$\overrightarrow{BC}=(1-2)\hat{i}+(5-3)\hat{j}+(5-5)\hat{k}=-\hat{i}+2\hat{j}$$

Now,

$$\text{Area of } \Delta ABC=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$$

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&3\\-1&2&0\end{vmatrix}=\hat{i}(-6)-\hat{j}(3)+\hat{k}(2+2)=-6\hat{i}-3\hat{j}+4\hat{k}$$

$$\therefore\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$$

Therefore, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

10. Find the area of the parallelogram whose adjacent sides are determined by the

vector $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$ **.**

Solution:

Let us consider,

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Now, the adjacent sides are given as:

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$|\vec{a}\times\vec{b}|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Therefore, the area of the given parallelogram is $15\sqrt{2}$ square units.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

Explanation:

Given, $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}|=1$.

$$|\vec{a}\times\vec{b}|=1$$

$$\left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right|=1$$

$$|\vec{a}||\vec{b}||\sin\theta|=1$$

$$3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\sin\theta=\frac{1}{\sqrt{2}}$$

$$\theta=\frac{\pi}{4}$$

Thus, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

So, the correct answer is B.

12. Area of a rectangle having vertices A, B, C, and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ **and** $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ **respectively is**

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Solution:

Explanation:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overline{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overline{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overline{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k},\ \overline{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as.

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$$

We know that, the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Thus, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ square units.

So, the correct answer is C.

Miscellaneous Exercise

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of *x*-axis.

Solution:

Let us consider,

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Hence, for $\theta = 30°$ we have:

$$\vec{r} = \cos 30°\hat{i} + \sin 30°\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Therefore, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

2. Find the scalar components and magnitude of the vector joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Solution:

Firstly let us consider,

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be found out by:

$$\overrightarrow{PQ} = \text{Position vector of Q} - \text{Position vector of P}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Therefore, the scalar components and the magnitude of the vector joining the given points are respectively

$\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

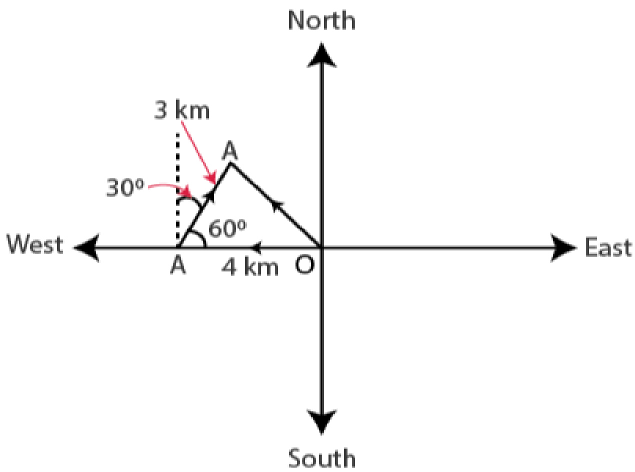
3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution:

It is given that,

Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



$\overrightarrow{OA} = -4\hat{i}$

$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60° + \hat{j}\left|\overrightarrow{AB}\right|\sin 60°$

$= \hat{i}3 \times \frac{1}{2} + \hat{j}3 \times \frac{\sqrt{3}}{2}$

$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

By the Triangle law of vector addition, we have

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$

$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

Therefore, the girl's displacement from her initial point of departure is

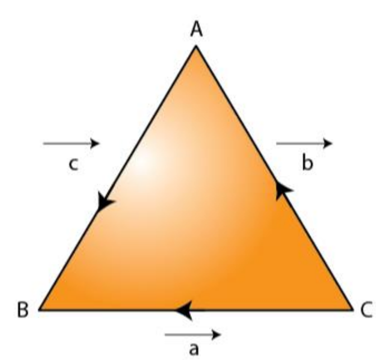
$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$.

4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution:

It is given that,

In ΔABC, let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ (as shown in the following figure).



So, by the Triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

And, we know that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$

Therefore, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$.

5. Find the value of *x* for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution:

We know,

Given $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

So, $|x(\hat{i} + \hat{j} + \hat{k})| = 1$.

Now,

$|x(\hat{i} + \hat{j} + \hat{k})| = 1$

$\sqrt{x^2 + x^2 + x^2} = 1$

$\sqrt{3x^2} = 1$

$\sqrt{3}\,x = 1$

$x = \pm \frac{1}{\sqrt{3}}$

Therefore, the required value of x is $\pm \frac{1}{\sqrt{3}}$

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution:

Let us consider the,

Given vectors,
$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$.

Then,
$$\vec{c} = \vec{a} + \vec{b} = (2+1)\hat{i} + (3-2)\hat{j} + (-1+1)\hat{k} = 3\hat{i} + \hat{j}$$
$$|\vec{c}| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$
$$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{i} + \hat{j})}{\sqrt{10}}$$

Therefore, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is
$$\pm 5 \cdot \hat{c} = \pm 5 \cdot \frac{1}{\sqrt{10}}(3\hat{i} + \hat{j}) = \pm\frac{3\sqrt{10}\hat{i}}{2} \pm \frac{\sqrt{10}}{2}\hat{j}.$$

7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Solution:

Let us consider the given vectors,

Given,
$\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$
$$2\vec{a} - \vec{b} + 3\vec{c} = 2(\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$$
$$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$$
$$= 3\hat{i} - 3\hat{j} + 2\hat{k}$$
$$|2\vec{a} - \vec{b} + 3\vec{c}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9+9+4} = \sqrt{22}$$

Therefore, the unit vector along $2\vec{a} - \vec{b} + 3\vec{c}$ is
$$\frac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|} = \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}.$$

8. Show that the points A (1, –2, –8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Solution:

Firstly let us consider,

Given points are: A (1, –2, –8), B (5, 0, –2), and C (11, 3, 7).

$\therefore \overrightarrow{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\overrightarrow{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\overrightarrow{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

$|\overrightarrow{AB}| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$

$|\overrightarrow{BC}| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$

$|\overrightarrow{AC}| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Therefore, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda : 1$. So, we have:

$$\overrightarrow{OB} = \frac{\lambda\overrightarrow{OC} + \overrightarrow{OA}}{(\lambda+1)}$$

$$5\hat{i} - 2\hat{k} = \frac{\lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})}{\lambda+1}$$

$$(\lambda+1)(5\hat{i} - 2\hat{k}) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$5(\lambda+1)\hat{i} - 2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i} + (3\lambda-2)\hat{j} + (7\lambda-8)\hat{k}$$

On equating the corresponding components, we have:

$$5(\lambda+1) = 11\lambda+1$$
$$5\lambda+5 = 11\lambda+1$$
$$6\lambda = 4$$
$$\lambda = \frac{4}{6} = \frac{2}{3}$$

Therefore, point B divides AC in the ratio $2:3$.

9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a}+\vec{b})$ and $(\vec{a}-3\vec{b})$ externally in the ratio 1: 2. Also, show that P is the midpoint of the line segment RQ.

Solution:

We know,

Given, $\overrightarrow{OP} = 2\vec{a} + \vec{b}$, $\overrightarrow{OQ} = \vec{a} - 3\vec{b}$.
Also, given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. So, on using the section formula, we have

$$\overrightarrow{OR} = \frac{2(2\vec{a}+\vec{b})-(\vec{a}-3\vec{b})}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1} = 3\vec{a}+5\vec{b}$$

Hence, the position vector of point R is $3\vec{a}+5\vec{b}$.
Now,

Position vector of the mid-point of RQ $= \dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$= \frac{(\vec{a}-3\vec{b})+(3\vec{a}+5\vec{b})}{2}$$

$$= 2\vec{a}+\vec{b}$$

$$= \overrightarrow{OP}$$

Therefore, P is the mid-point of the line segment RQ.

10. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ **and** $\hat{i}-2\hat{j}-3\hat{k}$**.**
Find the unit vector parallel to its diagonal. Also, find its area.
Solution:
Firstly let us consider,

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}-3\hat{k}$

We know that, the diagonal of a parallelogram is given by $\vec{a}+\vec{b}$.

$$\vec{a}+\vec{b} = (2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k} = 3\hat{i}-6\hat{j}+2\hat{k}$$

Hence, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{7} = \frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}.$$

So, the area of parallelogram ABCD $= |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$
$$= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$
$$= 22\hat{i} + 11\hat{j}$$
$$= 11(2\hat{i} + \hat{j})$$
$$\therefore |\vec{a} \times \vec{b}| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

Therefore, the area of the parallelogram is $11\sqrt{5}$ square units.

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution:

Firstly,

Let's assume a vector to be equally inclined to axes OX, OY, and OZ at angle α.

Then, the direction cosines of the vector are $\cos\alpha$, $\cos\alpha$, and $\cos\alpha$.

Now, we know that

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$
$$3\cos^2\alpha = 1$$
$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Therefore, the direction cosines of the vector which are equally inclined to the axes are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Hence proved.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d} = 15$.

Solution:

Assume,

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

As $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$$\vec{d} \cdot \vec{a} = 0$$
$$d_1 + 4d_2 + 2d_3 = 0 \qquad ...(i)$$

And,

$\vec{d}\cdot\vec{b}=0$

$3d_1-2d_2+7d_3=0 \qquad ...(ii)$

Also, given that:

$\vec{c}\cdot\vec{d}=15$

$2d_1-d_2+4d_3=15 \qquad ...(iii)$

On solving (i), (ii), and (iii), we obtain

$d_1=\frac{160}{3}, d_2=-\frac{5}{3} \text{ and } d_3=-\frac{70}{3}$

$\therefore \vec{d}=\frac{160}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$

Therefore, the required vector is $\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$

13. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ .

Solution:

Let's consider the

Sum of the given vectors is given by,

$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$

$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

Hence, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $\left(\hat{i}+\hat{j}+\hat{k}\right)$ with this unit vector is 1.

$$\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\sqrt{\lambda^2+4\lambda+44}=\lambda+6$$

$$\lambda^2+4\lambda+44=(\lambda+6)^2$$

$$\lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$

$$8\lambda=8$$

$$\lambda=1$$

Therefore, the value of A is 1.

14. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$.

Solution:

Lets assume,

As $\vec{a},\vec{b}$, and $\vec{c}$ are mutually perpendicular vectors, we have

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0.$

Given that:

$|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b}$, and $\vec{c}$ at angles $\theta_1,\ \theta_2$, and θ_3 respectively.

So, we have

$$\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}\qquad\left[\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}=0\right]$$

$$=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}$$

$$=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}\qquad\left[\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=0\right]$$

$$=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3=\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$$

$$=\frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}\qquad\left[\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=0\right]$$

$$=\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now, as $|\vec{a}|=|\vec{b}|=|\vec{c}|$, $\cos\theta_1=\cos\theta_2=\cos\theta_3$.

$\therefore \theta_1=\theta_2=\theta_3$

Therefore, the vector $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to $\vec{a},\vec{b}$, and $\vec{c}$.

Hence proved.

15. Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a}\neq\vec{0}, \vec{b}\neq\vec{0}$.

Solution:

It is given that

Required to prove:

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$

$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2$ [Distributivity of scalar products over addition]

$|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$ [$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Scalar product is commutative)]

$2\vec{a}\cdot\vec{b}=0$

$\vec{a}\cdot\vec{b}=0$ [$\vec{a}\neq\vec{0}, \vec{b}\neq\vec{0}$ (Given)]

Therefore, $\vec{a}$ and $\vec{b}$ are perpendicular.

Hence proved.

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b}\geq 0$ only when

(A) $0<\theta<\frac{\pi}{2}$ **(B)** $0\leq\theta\leq\frac{\pi}{2}$

(C) $0<\theta<\pi$ **(D)** $0\leq\theta\leq\pi$

Solution:

Explanation:

Let's assume θ to be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive

We also know, $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

So,

$\vec{a}\cdot\vec{b}\geq 0$

$|\vec{a}||\vec{b}|\cos\theta\geq 0$

$\cos\theta\geq 0$ [$|\vec{a}|$ and $|\vec{b}|$ are positive]

$0\leq\theta\leq\frac{\pi}{2}$

Therefore, $\vec{a}.\vec{b}\geq 0$ when $0\leq\theta\leq\frac{\pi}{2}$.

The correct answer is B.

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ **(B)** $\theta = \frac{\pi}{3}$ **(C)** $\theta = \frac{\pi}{2}$ **(D)** $\theta = \frac{2\pi}{3}$

Solution:
Explanation:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{b}| = 1$.

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$.

$$|\vec{a}+\vec{b}| = 1$$
$$(\vec{a}+\vec{b})^2 = 1$$
$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$$
$$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$$
$$|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$$
$$1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$
$$1 + 2.1.1\cos\theta + 1 = 1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}$$

Therefore, $\vec{a}+\vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$.

Hence the correct answer is D.

18. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ **is**

(A) 0 **(B) –1** **(C) 1** **(D) 3**

Solution:
Explanation:

It is given that,

$$\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$$
$$=\hat{i}\cdot\hat{i}+\hat{j}\cdot(-\hat{j})+\hat{k}\cdot\hat{k}$$
$$=1-\hat{j}\cdot\hat{j}+1$$
$$=1-1+1$$
$$=1$$

Hence the correct answer is C.

19. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Solution:

Explanation:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$$|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$$

$$|\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$$

$$\cos\theta = \sin\theta \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}$$

Thus, $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to $\frac{\pi}{4}$

So, the correct answer is B.